AN INTRODUCTION
TO THE
THEORY OF ELLIPTIC FUNCTIONS
AND HIGHER TRANSCENDENTALS

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BY

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PREFACE

In this book are reproduced the six public lectures which I delivered at the Calcutta University in 1925. The Appendices A and B are intended to add to the usefulness of the book as an introduction to the theory of elliptic functions and higher transcendentals, and it is earnestly hoped that the publication of this book may popularise the study of elliptic functions among advanced students of Higher Mathematics.

I take this opportunity to express my gratitude to my colleague, Dr. Bibhutibhusan Datta, who read all the proof-sheets and whose ungrudging help has considerably improved the printing of the book.

CALCUTTA,
April 16, 1928.

GANESH PRASAD.

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FIRST LECTURE

Elliptic Integrals.

Colleagues, students and other gentlemen!

It is a source of great pleasure to me to find in this hall such a large gathering of brilliant mathematical scholars. I hope and trust that my lectures will inspire some of them with enthusiasm for the subject of Elliptic Functions about which, borrowing the words of one of the greatest living leaders of mathematical thought, I may say: "Not even a century has yet elapsed since the elliptic functions were first introduced in Mathematics. From that time on the theory has increased to such an extent that now-a-days scarcely any other field of Mathematics can offer such an abundance of formal results and such a wealth of applications to different branches of the exact sciences. Moreover, the prophetic divination of Euler has become a reality, the discovery of this theory has essentially extended the bounds of mathematical analysis. New fields have been opened for mathematical thought and the number of fundamental ideas with which Mathematics operates has been vastly increased. A careful analysis of these fundamental ideas has formed the point of departure of a great number of investigations, the results of which form the peaks of our present day knowledge in Mathematics."

1. What we call an elliptic function may be briefly defined as the inverse function of an indefinite integral of the form

$$\int_0^x \mathrm{R}(x, w)\,dx$$

where w is $\sqrt{\mathrm{A}x^4+\mathrm{B}x^3+\mathrm{C}x^2+\mathrm{D}x+\mathrm{E}}$, A and B are not both zero, and $\mathrm{R}(x, w)$ is any rational algebraic function of x and w. The integral is now called an elliptic integral, although in the writings of Legendre it is designated "elliptic function."

2. The early history of elliptic integrals may be said to date from 1655 when Wallis considered the question of finding the length of an arc of an ellipse. The example of Wallis was followed by a number of distinguished mathematicians, including Jacob Bernoulli, Johann Bernoulli and Count Fagnano, who were led to elliptic integrals in connection with the rectification of various curves. Fagnano (1682-1766) began his investigation in 1714 and his earliest published paper of the year 1715 contains a remarkably simple solution of the problem of the halving of the quadrant of a lemniscate which I proceed to give to you on account of its historical importance. Fagnano was justly proud of his solution and left instructions that on his grave a lemniscate should be shown inscribed in recognition of his achievement.

Solution of Fagnano:

Let $(x^2+y^2)^2=2a^2(x^2-y^2)$ be the curve and let O be the origin and A be the other point in which the quadrant cuts the axis of x.

The problem is to find a point T such that

$$\text{arc OT}=\text{arc AT}.$$

$$\text{Let } \frac{x}{a}=\sqrt{t+t^2},\ \frac{y}{a}=\sqrt{t-t^2}.$$

Then, if S corresponds to $t=m$,

$$\text{arc OS} = \frac{a}{\sqrt{2}}\int_0^{t=m}\frac{dt}{\sqrt{t(1-t^2)}}.$$

Putting $t=\frac{1-v}{1+v}$, we have

$$\int_0^m \frac{dt}{\sqrt{t(1-t^2)}}=\int_{\frac{1-m}{1+m}}^1 \frac{dv}{\sqrt{v(1-v^2)}}=\int_0^1 - \int_0^{\frac{1-m}{1+m}}$$

Therefore, if M corresponds to $t=\frac{1-m}{1+m}$,

we have

$$\text{arc OS}=\text{arc OA}-\text{arc OM}=\text{arc AM}.$$

$$\text{If} \quad m=\frac{1-m}{1+m},$$

then M and S coincide, and we get the required point T.

3. The results of Fagnano's researches were collected and published by him in 1750 in his book entitled "Produzioni Matematiche." The early history of elliptic integrals ends with 1750. If I were asked to give the subsequent history of elliptic integrals and allied transcendentals in six words, I could do no better than mention the names of L. Euler (1707-1783), A. M. Legendre (1752-1833), N. H. Abel (1802-1829), C. G. J. Jacobi (1804-1851), Riemann (1826-1866) and K. T. W. Weierstrass (1815-1897).

4. Fagnano sent a copy of his book to the Royal Academy of Sciences of Berlin which forwarded it to Euler on the 23rd December, 1751, for consideration and report. Jacobi called that day the birth day of elliptic functions. As early as 1698, Johann Bernoulli had noticed that

$$f(x)dx \pm f(y)dy = 0$$

had as an integral an algebraic relation between x, y even if

$$\int f(x)dx$$

was a logarithm or an inverse trigonometric function, and he had formulated the question whether that property might hold for other transcendentals than the logarithm and the inverse trigonometric functions. Fagnano was the first to prove that that property was found in at least certain classes of elliptic integrals. For example, Fagnano showed that the equation

$$\frac{dx}{\sqrt{a+bx^2+cx^4}}+\frac{dy}{\sqrt{a+by^2+cy^4}}=0$$

was satisfied by an algebraic relation between x and y. The study of Fagnano's book encouraged Euler to study the subject of elliptic integrals and, on the 27th Jan., 1752, he came out with his first paper on the subject, which is on the integral $\int\frac{dx}{\sqrt{1-x^4}}$ and relates to Fagnano's investigations on the lemniscate. Next year he gave the general integral of the differential equation

$$\frac{dx}{\sqrt{1-x^4}}+\frac{dy}{\sqrt{1-y^4}}=0$$

(see *Nova Comm. Petropolitana,* Vol. 6) in the form

$$x=\frac{y\sqrt{1-c^4}+c\sqrt{1-y^4}}{1+c^2y^2},$$

c being a constant, and was thus led to the addition theorem for the integral $\int\frac{dx}{\sqrt{1-x^4}}$:

$$\text{If}\quad \int_0^x\frac{dx}{\sqrt{1-x^4}}+\int_0^y\frac{dy}{\sqrt{1-y^4}}=\int_0^z\frac{dz}{\sqrt{1-z^4}},$$

then z is an algebraic symmetric function, of x and y, namely,

$$z=\frac{y\sqrt{1-x^4}+x\sqrt{1-y^4}}{1+x^2y^2}.$$

5. In the year 1767, Euler published the complete integral of the general equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

where $X=Ax^4+Bx^3+Cx^2+Dx+E$ and $Y=Ay^4+By^3+Cy^2+Dy+E$.

The following method is a modification by Euler of the elegant procedure given by Lagrange in 1768 (*Misc. Taurinensia*, Vol. 4).

Take x and y to be functions of a variable t, then the above equation is equivalent to

$$\frac{dx}{dt}=\sqrt{X},\ \frac{dy}{dt}=-\sqrt{Y}\ .$$

Put $x+y=p,\ x-y=q$; then

$$\frac{dp}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=\sqrt{X}-\sqrt{Y},$$

$$\frac{dq}{dt}=\sqrt{X}+\sqrt{Y}.$$

$$\frac{d^2p}{dt^2}=\frac{X'}{2\sqrt{X}}\cdot\frac{dx}{dt}-\frac{Y'}{2\sqrt{Y}}\cdot\frac{dy}{dt}\ \text{(the dash denoting differentiation)}$$

$$=\frac{X'+Y'}{2}=D+C(x+y)+\frac{3}{2}B(x^2+y^2)+2A(x^3+y^3).$$

Now $$\frac{dp}{dt}\cdot\frac{dq}{dt}=X-Y=(x-y)\ [D+C(x+y)+B(x^2+xy+y^2)+A(x+y)(x^2+y^2)].$$

Therefore $$(x-y)\frac{d^2p}{dt^2}-\frac{dp}{dt}\frac{dq}{dt}=(x-y)^3[\tfrac{1}{2}B+A(x+y)]$$

i.e., $$q\frac{d^2p}{dt^2}-\frac{dp}{dt}\frac{dq}{dt}=q^3(\tfrac{1}{2}B+Ap)$$

i.e., $$\frac{2d^2p}{q^2dt^2}-\frac{2}{q^3}\frac{dq}{dt}\frac{dp}{dt}=B+2Ap$$

i.e., $$\frac{2d}{qdt}\left(\frac{1}{q}\frac{dp}{dt}\right)=B+2Ap$$

i.e., $$\frac{d}{dt}\left(\frac{1}{q}\frac{dp}{dt}\right)^2=B\frac{dp}{dt}+A\frac{d\,(p^2)}{dt}$$

$$\therefore\qquad\left(\frac{1}{q}\frac{dp}{dt}\right)^2=Bp+Ap^2+\text{constant (say F)}$$

i.e., $$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=B(x+y)+A(x+y)^2+F.$$

6. I propose to proceed to apply the above complete integral to determine the addition theorem for the function $\wp(u)$, given by

$$u=\int_{s=\wp(u)}^{\infty}\frac{ds}{\sqrt{4s^3-g_2s-g_3}}.$$

Consider $$\frac{ds}{\sqrt{4s^3-g_2s-g_3}}+\frac{ds_1}{\sqrt{4s_1{}^3-g_2s_1-g_3}}=0.$$

Put $A=0$, $B=4$, $C=0$, $D=-g_2$, $E=-g_3$, in the complete integral of the preceding article. Therefore the complete integral of the above equation is

$$\left(\frac{\sqrt{4s^3-g_2s-g_3}-\sqrt{4s_1{}^3-g_2s_1-g_3}}{s-s_1}\right)^2-4(s+s_1)=F$$

Since $\frac{ds}{\sqrt{4s^3-g_2s-g_3}}=-du$, for s being put $\wp(u)$;

and similarly $\frac{ds_1}{\sqrt{4s_1{}^3-g_2s_1-g_3}}=-d\mathrm{U}$, for s_1 being put $\wp(\mathrm{U})$;

we have $\mathrm{U}+u=$ a constant, say a.

Thus $\wp(u)=s$, $\wp(a-u)=s_1$,

Therefore $$\left\{\frac{\wp'(u)-\wp'(a-u)}{\wp(u)-\wp(a-u)}\right\}^2-4\{\wp(u)+\wp(a-u)\}=\mathrm{F} \quad \dots \quad (1)$$

Now $$u=\int_{\wp(u)}^{\infty}\frac{ds}{2s^{\frac{3}{2}}\sqrt{1-\frac{g_2}{4s^2}-\frac{g_3}{4s^3}}}$$

Therefore $$u=\int_{\wp(u)}^{\infty}\frac{ds}{2s^{\frac{3}{2}}}\left\{1+\tfrac{1}{2}\left(\frac{g_2}{4s^2}+\frac{g_3}{4s^3}\right)+\dots\right\}$$
$$=1/\sqrt{\wp(u)}+\text{lower powers of }\wp(u)$$

Therefore when $u=0$, $\wp(u)=\infty$.

Hence $\wp(u)\sim\frac{1}{u^2}$,

and $\wp'(u)\sim-\frac{2}{u^3}$.

Therefore, writing (1) in the form

$$\begin{aligned}&[4\wp^3(u)-g_2\wp(u)-g_3+4\wp^3(a-u)-g_2\wp(a-u)-g_3\\&\qquad-4\{\wp(u)+\wp(a-u)\}\{(\wp^2(u)+\wp^2(a-u)-2\wp(u)\wp(a-u)\}\\&\qquad\qquad-2\wp'(u)\wp'(a-u)]\\&\qquad\qquad\div\{\wp(u)-\wp(a-u)\}^2=\mathrm{F}\end{aligned}$$

i.e., $$\frac{4\{\wp(u)+\wp(a-u)\}\{\wp(u)\wp(a-u)-\frac{g_2}{4}\}-2g_3-2\wp'(u)\wp'(a-u)}{\{\wp(u)-\wp(a-u)\}^2}=\mathrm{F}$$

i.e.,

$$\frac{4\left\{1+\frac{\wp(a-u)}{\wp(u)}\right\}\left\{\wp(a-u)-\frac{g_2}{4\wp(u)}\right\}-\frac{2g_3}{\wp^2(u)}-\frac{2\wp'(u)\wp'(a-u)}{\wp^2(u)}}{\left\{1-\frac{\wp(a-u)}{\wp(u)}\right\}^2}=\mathrm{F};$$

and taking the limit for $u=0$, we have

$$\mathrm{F}=4\wp(a).$$

Therefore

$$\tfrac{1}{4}\left\{\frac{\wp'(u)-\wp'(a-u)}{\wp(u)-\wp(a-u)}\right\}^2-\wp(u)-\wp(a-u)=\wp(a).$$

Put $a=u+v$, then we have the addition theorem

$$\wp(u+v)=\tfrac{1}{4}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}^2-\wp(u)-\wp(v).$$

7. Euler obtained the complete integral of the equation

$$\frac{m\,dx}{\sqrt{1-x^4}}=\frac{n\,dy}{\sqrt{1-y^4}}$$

and showed that it was expressible as an algebraic function of x, y and an arbitrary constant. Thus Euler gave the fundamental ideas concerning the addition and multiplication of elliptic integrals of the first kind.

8. I propose now to introduce to you the transformation theory.

One method of solving the equation

$$\frac{dx}{\sqrt{\mathrm{X}}}=\pm\frac{dy}{\sqrt{\mathrm{Y}}}$$

is to assume beforehand an algebraic relation between x and y and determine the coefficients in that relation so that the above equation may be satisfied. It is easy to see that a linear relation, leaving the trivial case of $x=y$, is impossible. Euler assumed the next relation in order of simplicity and put

$$(ax^2+2bx+c)y^2+2(a'x^2+2b'x+c')y+a''x^2+2b''x+c''=0$$

and determined all the constants with the exception of one, which would be arbitrary, in order that the equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$$

might be satisfied. This form of the complete integral was used by him to deduce the addition theorem :

If $$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}+\int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} .$$

then the upper limit z is connected with x and y by the algebraic relation

$$z=\frac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2} .$$

9. Before mentioning the general transformation problem of Jacobi, I wish to give, on account of their historical importance, the transformations of Gauss and Landen. According to Gauss,

$$\frac{dx}{\sqrt{(1-p^2x^2)(1-q^2x^2)}}=\frac{dy}{\sqrt{[1-(\frac{1}{2}p_1)^2y^2][1-(\frac{1}{2}q_1)^2y^2]}}$$

when $$x=\frac{y}{1+\frac{1}{4}q^2y^2},\; p_1=p+\sqrt{p^2-q^2},\; q_1=p-\sqrt{p^2-q^2};$$

according to Landen and Lagrange (*Phil. Trans.* 1771), the transformation

$$x=y\sqrt{\frac{1-\{\frac{1}{2}(p+q)\}^2y^2}{1-pqy^2}}$$

changes $$\frac{dx}{\sqrt{(1-p^2x^2)(1-q^2x^2)}} \text{ into } \frac{dy}{\sqrt{(1-p_1^2y^2)(1-q_1^2y^2)}}$$

where $p_1 = \frac{1}{2}(p+q)$, $q_1 = \sqrt{pq}$.

10. Calling $y = \dfrac{a_0 + a_1 x + a_2 x^2 + \ \ldots \ + \ a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ \ldots \ + \ b_m x^m}$

a transformation of the m^{th} order, Jacobi showed that, by suitably choosing the constant coefficients, the transformation changes

$$\frac{dy}{\sqrt{A_1 + B_1 y + C_1 y^2 + D_1 y^3 + E_1 y^4}}$$

$$\text{into} \qquad \frac{1}{M} \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}$$

where M is a constant. It follows that two transformations of orders m and m_1 are equivalent to a single transformation of order mm_1, and that all transformations admit of being made up out of the transformation of the 2nd order and those transformations which correspond to prime numbers. Whilst Jacobi restricted himself to the consideration of a special type of relation between y and x, Abel's transformation theory dealt with the solution of the more general case where x and y are connected by any algebraic relation, rational or irrational.

If

$$\int \frac{dy}{\sqrt{(1-y^2)(1-l^2y^2)}} = \frac{1}{M} \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

then the relation * between k and l is called a *modular equation* and the relation between M and k is called a *multiplicator equation.*

---

* The transformation problem for linear transformations was first solved by Abel (*Crelle's Journal*, Vols. 3 and 4) who gave all the six solutions which are

(1) $l = \pm k$, $y = \pm x$, $y = \pm \frac{1}{kx}$, $\frac{1}{M} = \pm 1$.

(2) $l = \pm \frac{1}{k}$, $y = \pm \frac{x}{k}$, $y = \pm \frac{1}{x}$. $\frac{1}{M} = \pm k$.

(3) $l = \pm \left(\frac{1 - \sqrt{k}}{1 + \sqrt{k}} \right)^2$, $y = \pm \frac{1 + \sqrt{k}}{1 - \sqrt{k}} \frac{1 \pm x\sqrt{k}}{1 \mp x\sqrt{k}}$, $2M = \pm i\,(1 + \sqrt{k}\,)^2$.

(4) $l = \pm \left(\frac{1 + \sqrt{k}}{1 - \sqrt{k}} \right)^2$, $y = \pm \frac{1 - \sqrt{k}}{1 + \sqrt{k}} \cdot \frac{1 \pm x\sqrt{k}}{1 \mp x\sqrt{k}}$, $2M = \pm i(1 - \sqrt{k}\,)^2$.

Denoting $\sqrt[4]{k}$ by u and $\sqrt[4]{l}$ by v, the modular equations for the transformations of the 3rd, 5th, 7th and 11th orders are respectively

$$v^4+2v^3u^3-2vu-u^4=0,$$

$$v^6-u^6-4uv(1-u^4v^4)-5u^2v^2(u^2-v^2)=0,$$

$$v^8-8u^7v^7+28u^6v^6-56u^5v^5+70u^4v^4-56u^3v^3+28u^2v^2-8uv+u^8=0,$$

and

$$v^{12}-v^{11}u^3(22-32u^8)+44v^{10}u^6+22v^9u(1+4u^8)+165v^8u^4$$
$$+132u^7v^7+44u^2v^6(1-u^8)-132u^5v^5-165v^4u^8-22u^3v^3(4+u^8)$$
$$-44v^2u^6-uv(32-22u^8)-u^{12}=0.$$

The multiplicator equation for the transformation of the 5th order is

$$(Z-1)^6-4(Z-1)^5+256k^2(1-k^2)(Z-1)+256k^2(1-k^2)=0,$$

$$Z \text{ standing for } \frac{1}{M}$$

11. When Legendre took up in 1786 the consideration of the theory of elliptic functions, Euler's addition theorem and the transformation theorem of Landen and Lagrange were the two fundamental ideas which the theory then contained. After working for nearly forty years, Legendre collected his results in his famous book " Traité des fonctions elliptiques et des intégrales eulériennes " (Paris, 2 Vols. and three supplements, 1825-1828). Writing of Legendre's achievement, in 1852, Dirichlet said, " It is the unforgettable distinction of Legendre to have recognised in the discoveries of Fagnano, Euler, Landen, and Lagrange the seed of an important branch of Analysis and to have built up on these foundations by the work of half a lifetime an independent theory which comprehends all integrals in which no other irrationality occurs than a square root and in which the variable does not go beyond the fourth power."

$$(5)\ l=\pm\left(\frac{1-\sqrt{-k}}{1+\sqrt{-k}}\right)^2,\ y=\pm\frac{1+\sqrt{-k}}{1-\sqrt{-k}}\cdot\frac{1\pm x\sqrt{-k}}{1\mp x\sqrt{-k}},\ 2M=\pm i(1+\sqrt{-k})^2.$$

$$(6)\ l=\pm\left(\frac{1+\sqrt{-k}}{1-\sqrt{-k}}\right)^2,\ y=\pm\frac{1-\sqrt{-k}}{1+\sqrt{-k}}\cdot\frac{1\pm x\sqrt{-k}}{1\mp x\sqrt{-k}},\ 2M=\pm i(1-\sqrt{-k})^2.$$

12. Legendre first reduces the general elliptic integral to the two forms

$$\int \frac{dx}{(A+Bx+Cx^2)\sqrt{f(x)}}, \quad \int \frac{dx}{(1+nx)\sqrt{f(x)}},$$

$f(x)$ being the general polynomial of the fourth degree ; then he gets rid of the odd powers of x from $f(x)$ first by the use of the substitution

$$y=\sqrt{\frac{(x-x_1)(x-x_2)}{(x-x_3)(x-x_4)}},$$

x_1, x_2, x_3, x_4 being the four roots of $f(x)=0$,

and secondly by the linear substitution

$$x=\frac{p+qy}{1+y}.$$

Next he puts

$$x^2=\frac{A_1+B_1\sin^2\phi}{C_1+D_1\sin^2\phi}$$

and then changes

$$\frac{dx}{\sqrt{f(x)}}$$

into the form

$$\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}},$$

leaving aside a constant multiplier. The final result of Legendre is, that the most general elliptic integral must reduce to one of the three canonical forms:

$$\int_0^\phi \frac{d\phi}{\sqrt{1-c^2\sin^2\phi}}, \int_0^\phi d\phi\sqrt{1-c^2\sin^2\phi},$$

$$\int_0^\phi \frac{1}{(1+n\sin^2\phi)} \frac{d\phi}{\sqrt{1-c^2\sin^2\phi}}.$$

These are called by him the normal integrals of the first, second and third kinds and are denoted respectively by $F(c, \phi)$. $E(c, \phi)$ and $\Pi(n, c, \phi)$. Legendre submitted each of the three kinds of integrals to a careful investigation and discovered many of its most important properties. According to Dirichlet, it was only on account of his remarkable perseverance that Legendre made repeated efforts and overcame difficulties which in the then state of Analysis were almost insurmountable.

13. An elliptic integral which reduces to an integral of rational functions is called a *pseudo-elliptic integral* and the question of the conditions under which such a reduction takes place is of interest. This question was considered by Abel (*Crelle's Journal*, Vol. 1) with reference to integrals of the form

$$\int\frac{f(x)\,dx}{\sqrt{R(x)}},$$

where f and R are polynomials in x and he proved that there exist particular linear functions $x-a$ for which

$$\int\frac{(x-a)dx}{\sqrt{R(x)}}$$

is a pseudo-elliptic integral if the continued fraction for $\sqrt{R(x)}$ is periodic and the period shows symmetry.*

Degenerate forms of the elliptic integral

$$\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}}$$

are obtained when two of the quantities α, β, γ, δ are equal; the integral becoming equal in that case to inverse circular or hyperbolic functions.

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* Examples.

$$(1)\quad\int\frac{3x+a}{\sqrt{R}}dx=\log\frac{x^2+ax+\sqrt{R}}{x^2+ax-\sqrt{R}}$$

where $\quad R=(x^2+ax)^2+ex.$

$$(2)\quad\int\frac{4x+a}{\sqrt{R}}dx=\log\frac{x^2+ax+b+\sqrt{R}}{x^2+ax+b-\sqrt{R}}+\tfrac{1}{2}\log\frac{x^2+ax-b+\sqrt{R}}{x^2+ax-b-\sqrt{R}},$$

where $\quad R=(x^2+ax+b)^2-4abx.$

SECOND LECTURE.

THE OLD THEORY OF ELLIPTIC FUNCTIONS

13. The old theory of elliptic functions was built up chiefly by the labours of Abel and Jacobi and may be easily distinguished from the modern theory by the absence of the general ideas, based on the theory of functions of a complex variable, which we owe to Liouville, Riemann, Weierstrass and Mittag-Leffler and which are closely interwoven in the fabric of the modern theory. But the old theory was an absolute departure from Legendre's ideas and contained as its corner stones the study of the inverse function of the elliptic integral and the discovery of the double periodicity of, that function. Both these fundamental points were taken up by the rivals, Abel and Jacobi, independently of each other.

14. Abel's work on the subject dates back to 1823. In a letter, which he wrote to his friend Holmboe on the 3rd August, 1823, Abel expressed the opinion that the right way to study the elliptic integral was to consider the inverse function. His researches on elliptic functions appeared mostly in the first five volumes of *Crelle's Journal* beginning with 1826, and some in *Astronomische Nachrichten*. The most important results were published by Abel in his memoirs "Recherches sur les fonctions elliptiques" (*Crelle's Journal*, Vols. 2 and 3, 1827 and 1828) and "Précis d'une théorie des fonctions elliptiques" (*Crelle's Journal*, Vol. 4, 1829). According to Koenigsberger, Abel was in possession of the principle of double periodicity as early as 1825.

15. Jacobi's first paper on the subject of elliptic functions dealt with the transformation problem and appeared in Sept., 1827 in the *Astronomische Nachrichten* with the title "Extraits de deux lettres de M. Jacobi de l'Université de Königsburg a l'editeur."

The second paper appeared in the same journal in December, 1827 and contained the symbols which became current later on, *viz.*, $\text{am}\, u=\phi$, $\sin\text{am}\, u=x$, $\cos\text{am}\, u=\sqrt{1-x^2}$, $\Delta\,\text{am}\, u=\sqrt{1-k^2x^2}$, where

$$u=\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

In 1829, soon after the death of Abel, Jacobi published his famous book, " Fundamenta Nova Theoriae Functionum Ellipticarum," which contains in a very terse form the majority of the most essential results of the theory of elliptic functions as known to Jacobi. Jacobi's subsequent researches on the subject appeared in *Crelle's Journal* and were intended to form a second part of the " Fundamenta." Jacobi left in his lectures a new foundation of the theory of elliptic functions, based on his theta-functions.

16. Although Abel's researches did not receive in his life-time that appreciation which they deserved, in the very brief period of his existence he did work which would continue to receive the careful attention of mathematicians for centuries. According to Professor Mittag-Leffler, Weierstrass was a great admirer of Abel's work and found much inspiration from it in his investigations on elliptic and Abelian functions. As regards Jacobi, Dirichlet wrote as follows : "Jacobi's scientific career covered exactly a quarter of a century and therefore a much shorter period than the careers of most of the earlier mathematicians of the first rank. It was scarcely half the length of the time over which Euler's activities had stretched, although, with Euler, Jacobi had a great resemblance, not only because of his versatility and productivity but also because he had at every moment at his finger's ends all the mathematical resources."

17. After this historical introduction, I propose to give you first the elements of the theory. As

$$u=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=-\int_0^{-\phi}\frac{d\phi_1}{\sqrt{1-k^2\sin^2\phi_1}}$$

we have $\phi=\text{am}\ u$, $-\phi=\text{am}\ (-u)$.

Thus $\text{am}\ (-u)=-\text{am}\ (u)$,

and, consequently,

$$\sin\text{am}\ (-u)=-\sin\text{am}\ (u),$$

$$\cos\text{am}\ (-u)=\cos\text{am}\ (u),$$

$$\Delta\ \text{am}\ (-u)=\Delta\ \text{am}\ (u);$$

also it follows that $\sin\text{am}\ (0)=0$, $\cos\text{am}\ (0)=1$, $\Delta\ \text{am}\ (0)=1$.

Further $$\frac{du}{d\phi} = \frac{1}{\sqrt{1-k^2\sin^2\phi}},$$

i.e., $$\frac{d \text{ am } u}{du} = \sqrt{1-k^2\sin^2\text{am}(u)} = \Delta \text{ am } u.$$

Hence $$\frac{d \sin \text{ am } u}{du} = \cos \text{ am } u.\ \Delta \text{ am } u,$$

$$\frac{d \cos \text{ am } u}{du} = -\sin \text{ am } u.\ \Delta \text{ am } u,$$

$$\frac{d\, \Delta \text{ am } u}{du} = -k^2 \sin \text{am } u.\ \cos \text{am } u.$$

Using the above results, we have the differential equation for sin am u :—

$$\frac{d^2y}{du^2} + y\{(1+k^2) - 2k^2y^2\} = 0,$$

where $y = \sin \text{ am } u$.

18. In 1838, Gudermann, a pupil of Jacobi, introduced the symbols sn, cn, and dn for sin am, cos am and Δ am. In what follows, I will use the symbols of Gudermann. But before proceeding to give you the addition-theorems for sn$u$, cn$u$ and dn$u$, I will explain a few ways of geometrically representing these functions.

(a) Take a circle of radius R and centre C, and take a point O inside it at a distance δ from C. Through O draw a chord POQ and mark on it a point p such that

$$\text{O}p = l\sqrt{\frac{2(\text{R}+\delta)}{\text{PQ}}},\ l \text{ being a constant.}$$

Also let AB be the diameter of the circle through O.

Then, denoting

$$\frac{\textit{area } p_0\text{OP}}{l^2} \text{ by } u,$$

where p_0 is the intersection of the locus of p with the diameter through O, and the $\angle$PCA by 2ϕ, we have

$$\phi = \text{am } u, \quad \frac{\text{OP}}{\text{R}+\delta} = \text{dn}u, \text{ and } k = \frac{2\sqrt{\text{R}\delta}}{\text{R}+\delta}.$$

(b) Draw the elastic curve with the line of tension as the x-axis. Then its equation is $y\rho = c^2$, where ρ is the radius of curvature at (x, y) and c is a constant.

If $OA=a$, $AP=s$, where O is the origin, A the point in which the axis of y cuts the curve, and P the point (x, y); then

$$\frac{y}{a}=\mathrm{cn}\,\frac{s}{c}.$$

(c) Take the quartic $(x^2+y^2)(x^2+b^2y^2)=a^4$,

and let S denote the area bounded by the axis of x, the curve and the radius vector making an angle ϕ with the axis of x. Then, taking u for

$$\frac{2S}{a^2},$$

we have

$$\phi=\mathrm{am}u,\ \frac{x}{r}=\mathrm{cn}u,\ \frac{y}{r}=\mathrm{sn}u,\ \frac{a^2}{r^2}=\mathrm{dn}u,\quad k=\sqrt{1-b^2}.$$

19. In my first lecture, I gave the addition theorem :—

$$\text{If}\ \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}+\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$\text{then}\quad z=\frac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2}.$$

Hence, denoting the three integrals by $u+v$, u and v respectively, we have

$$\mathrm{sn}(u+v)=\frac{\mathrm{sn}u.\ \mathrm{cn}v.\ \mathrm{dn}v.+\mathrm{sn}v.\ \mathrm{cn}u.\ \mathrm{dn}u.}{1-k^2\ \mathrm{sn}^2u\ \mathrm{sn}^2v}.$$

From this follow

$$\mathrm{cn}(u+v)=\frac{\mathrm{cn}u.\ \mathrm{cn}v-\mathrm{sn}u.\ \mathrm{sn}v.\ \mathrm{dn}u.\ \mathrm{dn}v}{1-k^2\mathrm{sn}^2u\ \mathrm{sn}^2v},$$

$$\mathrm{dn}(u+v)=\frac{\mathrm{dn}u.\ \mathrm{dn}v-k^2\mathrm{sn}u.\ \mathrm{sn}v.\ \mathrm{cn}u.\ \mathrm{cn}v}{1-k^2\ \mathrm{sn}^2u\ \mathrm{sn}^2v};$$

changing v into $-v$, and, remembering that $\mathrm{sn}(-v)=-\mathrm{sn}(v)$, $\mathrm{cn}(-v)=\mathrm{cn}(v)$, $\mathrm{dn}(-v)=\mathrm{dn}(v)$, we have

$$\mathrm{sn}(u-v)=\frac{\mathrm{sn}u\ \mathrm{cn}v\ \mathrm{dn}v-\mathrm{sn}v\ \mathrm{cn}u\ \mathrm{dn}u}{1-k^2\ \mathrm{sn}^2u\ \mathrm{sn}^2v}$$

with similar results for cn and dn.

20. As in the case of the circular functions, a large number of formulae may be deduced from the results given above.

For example,

$$\operatorname{sn}(2u)=\frac{2\operatorname{sn}u.\ \operatorname{cn}u.\ \operatorname{dn}u}{1-k^2\ \operatorname{sn}^2u},$$

$$\operatorname{sn}(u+v)+\operatorname{sn}(u-v)=\frac{2\operatorname{sn}u.\ \operatorname{cn}v.\ \operatorname{dn}v}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v},$$

$$\operatorname{sn}(u+v).\ \operatorname{sn}(u-v)=\frac{\operatorname{sn}^2u-\operatorname{sn}^2v}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v},$$

$$1+\operatorname{cn}(u+v)\ \operatorname{cn}(u-v)=\frac{\operatorname{cn}^2u+\operatorname{cn}^2v}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v},$$

$$1+\operatorname{dn}(u+v)\ \operatorname{dn}(u-v)=\frac{\operatorname{dn}^2u+\operatorname{dn}^2v}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v},$$

$$\operatorname{sn}(u\pm v)\ \operatorname{cn}(u\mp v)=\frac{\operatorname{sn}u.\ \operatorname{cn}u.\ \operatorname{dn}v\pm\operatorname{sn}v.\operatorname{cn}v.\operatorname{dn}u}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v}.$$

$$\operatorname{sn}(3u)=\frac{\operatorname{sn}u\{3-4(1+k^2)\ \operatorname{sn}^2u+6k^2\ \operatorname{sn}^4u-k^4\ \operatorname{sn}^8u\}}{1-6k^2\ \operatorname{sn}^4u+4k^2(1+k^2)\ \operatorname{sn}^6u-3k^4\ \operatorname{sn}^8u}.$$

21. In order to prove that the functions sn$u$, cn$u$, dn$u$ have each two periods, we may proceed as follows :—

(*a*) Let K stand for

$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\ \sin^2\phi}},$$

then sn K=1, cn K=0, dn K$=\sqrt{1-k^2}$.

Now by the addition theorem

$$\operatorname{sn}(u+v)=\frac{\operatorname{sn}u.\ \operatorname{cn}v.\ \operatorname{dn}v+\operatorname{sn}v.\ \operatorname{cn}u.\ \operatorname{dn}u}{1-k^2\ \operatorname{sn}^2u\ \operatorname{sn}^2v}.$$

Therefore, putting $v=\mathrm{K}$, we have

$$\operatorname{sn}(u+\mathrm{K})=\frac{\operatorname{cn}u\ \operatorname{dn}u}{1-k^2\ \operatorname{sn}^2u}=\frac{\operatorname{cn}u}{\operatorname{dn}u};$$

similarly

$$\mathrm{cn}(u+\mathrm{K})=-k'\,\frac{\mathrm{sn}u}{\mathrm{dn}u} \text{ and } \mathrm{dn}(u+\mathrm{K})=\frac{k'}{\mathrm{dn}u},$$

k' standing for $\sqrt{1-k^2}$.

Also $$\mathrm{sn}(u+\mathrm{K}+\mathrm{K})=\frac{\mathrm{cn}(u+\mathrm{K})}{\mathrm{dn}(u+\mathrm{K})}=-\mathrm{sn}\,u,$$

and, consequently, $\mathrm{sn}(u+4\mathrm{K})=\mathrm{sn}\,u$.

Thus it is proved that 4K is a period of sn u; similarly it can be proved that 4K and 2K are respectively periods of cn u and dn u.

(*b*) Let K′ stand for $\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}$. Now, since

$$u=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}$$

becomes

$$\frac{u}{i}=\int_0^{\psi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

on putting $\sin\phi=i\tan\psi$, we have

$$\mathrm{sn}\left(\frac{u}{i},k\right)=\sin\psi=-i\tan\phi=-i\,\frac{\mathrm{sn}(u,k')}{\mathrm{cn}(u,k')};$$

and, consequently,

$$\mathrm{sn}(iu,k)=\frac{i\,\mathrm{sn}(u,k')}{\mathrm{cn}(u,k')},$$

$$\mathrm{cn}(iu,k)=\frac{1}{\mathrm{cn}(u,k')},$$

$$\mathrm{dn}(iu,k)=\frac{\mathrm{dn}(u,k')}{\mathrm{cn}(u,k')}.$$

Therefore
$$\frac{1}{\mathrm{sn}(i\,\mathrm{K}', k)}=\frac{\mathrm{cn}(\mathrm{K}', k')}{i\,\mathrm{sn}(\mathrm{K}', k')}=0,$$

$$\frac{1}{\mathrm{cn}(i\mathrm{K}', k)}=0,$$

$$\frac{1}{\mathrm{dn}(i\mathrm{K}', k)}=0,$$

Also $\mathrm{sn}\,(2i\,\mathrm{K}', k)=0$, $\mathrm{cn}(2i\mathrm{K}', k)=-1$, $\mathrm{dn}(2i\,\mathrm{K}', k)=-1$.

Therefore, applying the addition theorem, we have

$$\mathrm{sn}\,(u+2i\,\mathrm{K}')=\mathrm{sn}\,u,$$

$$\mathrm{cn}\,(u+2i\mathrm{K}')=-\mathrm{cn}\,u,$$

$$\mathrm{dn}\,(u+2i\,\mathrm{K}')=-\mathrm{dn}\,u.$$

Thus $2i\mathrm{K}'$ is a period of $\mathrm{sn}\,u$ and $4i\,\mathrm{K}'$ is a period of both $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$. Also $2\mathrm{K}+2i\mathrm{K}'$ is a period of $\mathrm{cn}\,u$.

Therefore the following formulae hold, m and n being any integers:

$$\mathrm{sn}\,(u+4m\mathrm{K}+2ni\,\mathrm{K}')=\mathrm{sn}\,u,$$

$$\mathrm{cn}\,(u+4m\mathrm{K}+2n\,\overline{\mathrm{K}+i\mathrm{K}'})=\mathrm{cn}\,u,$$

$$\mathrm{dn}\,(u+2m\mathrm{K}+4ni\mathrm{K}')=\mathrm{dn}\,u.$$

22. The three functions become infinite at every point of the form

$$u=2m\mathrm{K}+(2n+1)i\mathrm{K}'\,;$$

the zeros of sn, cn and dn are respectively

$$2m\mathrm{K}+2ni\mathrm{K}',\ (2m+1)\mathrm{K}+2ni\mathrm{K}' \text{ and } (2m+1)\mathrm{K}+(2n+1)i\mathrm{K}'.$$

23. I proceed now to express *sn* in terms of the $\wp$-function which I introduced to you in my first lecture. We have

$$\{\wp'(u)\}^2=4\wp^3(u)-g_2\wp(u)-g_3$$

as
$$u=\int_{\wp(u)}^{\infty}\frac{ds}{\sqrt{4s^3-g_2s-g_3}}\,;$$

and
$$\{\mathrm{sn}'(u)\}^2=(1-\mathrm{sn}^2u)(1-k^2\mathrm{sn}^2u)$$

as
$$u=\int_0^{\mathrm{sn}(u)}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Now put
$$\mathrm{sn}^2u=\frac{a^2}{\wp\left(\frac{u}{a}\right)-\beta},$$

where a and β are two constants to be determined.

Remembering that $\frac{d}{du}(\mathrm{sn}^2u)=2\mathrm{sn}\,u\;\mathrm{cn}\,u\;\mathrm{dn}\,u$,

we have
$$2\mathrm{sn}u\;\mathrm{cn}u\;\mathrm{dn}u=\frac{-a^2}{\{\wp\left(\frac{u}{a}\right)-\beta\}^2}\times\frac{d}{du}\{\wp\left(\frac{u}{a}\right)\},$$

i.e.,
$$a^2\frac{d}{du}\wp\left(\frac{u}{a}\right)=-2\mathrm{sn}u\;\mathrm{cn}u\;\mathrm{dn}u\;\{\wp\left(\frac{u}{a}\right)-\beta\}^2,$$

and
$$a^2\left\{\frac{d}{du}\wp\left(\frac{u}{a}\right)\right\}^2=\frac{4}{a^2}\mathrm{sn}^2u\;\mathrm{cn}^2u\;\mathrm{dn}^2u\;\{\wp\left(\frac{u}{a}\right)-\beta\}^4$$
$$=4\{\wp\left(\frac{u}{a}\right)-\beta-a^2\}\{\wp\left(\frac{u}{a}\right)-\beta-k^2a^2\}$$
$$\times\{\wp\left(\frac{u}{a}\right)-\beta\};$$

putting v for $\frac{u}{a}$, we have
$$\{\wp'(v)\}^2=4\{\wp(v)-\beta-a^2\}\{\wp(v)-\beta-k^2a^2\}\{\wp(v)-\beta\}.$$

Denote the zeros of the right side by e_1, e_2, e_3 so that
$$\beta+a^2=e_1,\quad \beta+k^2a^2=e_2,\quad \beta=e_3.$$

Then, since $k^2<1$, $e_1>e_2>e_3$,
$$\text{and } a^2=e_1-e_3,\qquad k^2=\frac{e_2-e_3}{e_1-e_3}.$$

Now impose the conditions :
$$e_1+e_2+e_3=0,$$
$$e_1e_2+e_2e_3+e_3e_1=-\tfrac{1}{4}g_2,$$
$$e_1e_2e_3=\tfrac{1}{4}g_3\;;$$

then
$$\{\wp'(v)\}^2=4\wp^3(v)-g_2\wp(v)-g_3.$$

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Thus $$\operatorname{sn}^2(\sqrt{e_1-e_3}.u)=\frac{e_1-e_3}{\wp(u)-e_3},$$

and, conversely,

$$\wp(u)=e_3+\frac{e_1-e_3}{\operatorname{sn}^2(\sqrt{e_1-e_3}.u)}.$$

24. A large number of expressions for the elliptic functions were obtained by Abel and Jacobi in infinite series and infinite products. The following are a few of the most important of these expressions given by Jacobi.

(*a*) $$\operatorname{sn}\frac{2\mathrm{K}x}{\pi}=\frac{1}{\sqrt{k}}.\frac{2q^{\frac{1}{4}}\sin x\prod\limits_{n=1}^{\infty}(1-2q^{2n}\cos 2x+q^{4n})}{\prod\limits_{n=1}^{\infty}(1-2q^{2n-1}\cos 2x+q^{4n-2})},$$

$$\operatorname{cn}\frac{2\mathrm{K}x}{\pi}=\sqrt{\frac{k'}{k}}\;\frac{2q^{\frac{1}{4}}\cos x\prod\limits_{n=1}^{\infty}(1+2q^{2n}\cos 2x+q^{4n})}{\prod\limits_{n=1}^{\infty}(1-2q^{2n-1}\cos 2x+q^{4n-2})},$$

$$\operatorname{dn}\frac{2\mathrm{K}x}{\pi}=\sqrt{k'}\frac{\prod\limits_{n=1}^{\infty}(1+2q^{2n-1}\cos 2x+q^{4n-2})}{\prod\limits_{n=1}^{\infty}(1-2q^{2n-1}\cos 2x+q^{4n-2})},$$

where q stands for $e^{-\pi\mathrm{K}'/\mathrm{K}}$.

(*b*) Denoting

$$\prod_{n=1}^{\infty}\frac{(1-2q^{2n-1}\cos 2x+q^{4n-2})}{\prod\limits_{n=1}^{\infty}(1-q^{2n-1})^2}\quad\text{by}\quad\Theta\left(\frac{2\mathrm{K}x}{\pi}\right)\div\Theta(0)$$

where $\Theta(0)$ is a constant, and

$$\frac{2q^{\frac{1}{4}}\sin x\prod\limits_{n=1}^{\infty}(1-2q^{2n}\cos 2x+q^{4n})}{\prod\limits_{n=1}^{\infty}(1-q^{2n-1})^2}\quad\text{by}\quad\mathrm{H}\left(\frac{2\mathrm{K}x}{\pi}\right)\div\Theta(0),$$

Jacobi expressed the elliptic functions sn cn, dn in terms of Θ and H thus :—

$$\operatorname{sn}u=\frac{1}{\sqrt{k}}.\frac{\mathrm{H}(u)}{\Theta(u)},$$

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$$\operatorname{cn}u=\sqrt{\frac{k'}{k}}.\frac{\mathrm{H}(u+\mathrm{K})}{\Theta(u)},$$

$$\operatorname{dn}(u)=\sqrt{k'}.\frac{\Theta(u+\mathrm{K})}{\Theta(u)}.$$

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In the above

$$\frac{\Theta(0)}{\Theta(\mathrm{K})}=\sqrt{k'},$$

(c) **Jacobi's** trigonometric series for the elliptic functions sn, cn, dn are

$$\frac{2k\mathrm{K}}{\pi}\operatorname{sn}\left(\frac{2\mathrm{K}x}{\pi}\right)=\sum_{n=1}^{\infty}\frac{4q^{\frac{2n-1}{2}}}{1-q^{2n-1}}\sin(2n-1)x$$

and two similar series ; also

$$\Theta\left(\frac{2\mathrm{K}x}{\pi}\right)=1-2q\cos 2x+2q^{4}\cos 4x-2q^{9}\cos 6x+\ldots$$

$$\mathrm{H}\left(\frac{2\mathrm{K}x}{\pi}\right)=2q^{\frac{1}{4}}\sin x-2q^{\frac{9}{4}}\sin 3x+2q^{\frac{25}{4}}\sin 5x-2q^{\frac{49}{4}}\sin 7x+\ldots$$

25. In addition to the above two series which are generally denoted by $\theta(x)$ and $\theta_1(x)$, Jacobi introduced two more theta-functions, *viz.*,

$$\theta_2(x)=2q^{\frac{1}{4}}\cos x+2q^{\frac{9}{4}}\cos 3x+\cdots$$

and

$$\theta_3(x)=1+2q\cos 2x+2q^{4}\cos 4x+\ldots$$

The lectures of Jacobi on elliptic functions based on the theta-functions were written out by Borchardt in 1838 and mark not only a great progress in the theory of elliptic functions but also prepare the way for the development of the theory of Abelian functions in particular and multiply-periodic functions in general.

26. Although the Jacobian elliptic functions sn, cn, dn are being, or have been, discarded in favour of the Weierstrassian ℘-function, the theta-functions still hold the field. According to Poincaré, the simplicity of the development of these series, the rapidity of their convergence, the elegance of their properties assure to them a place of importance and from this place they cannot be dislodged. For the numerical calculation of the various other elliptic functions, the theta-functions are very useful.

THIRD LECTURE.

THE MODERN THEORY OF ELLIPTIC FUNCTIONS.

27. For a clear understanding of to-day's lecture and the next lecture, the following definitions will be useful :—

(1) If $f(z+2\Omega)=f(z)$, then $f(z)$ is called a *periodic function* and 2Ω is a *period* of $f(z)$.

(2) If all the periods of a function can be represented as positive or negative integral multiples of one and the same period 2ω, then the function is called *simply periodic*, and 2ω is called the *primitive period*.

(3) If a periodic function is not simply periodic and is not a constant, then it can be shown that it is possible to find two periods 2ω, $2\omega'$, in an infinite number of ways, such that all the remaining periods can be expressed in terms of these by expressions of the form $2m\omega+2n\omega'$, m and n being integral ; in this case the function is said to be *doubly periodic* and each system of two periods 2ω, $2\omega'$ having the aforesaid property is called a *primitive pair of periods*.

(4) If z_0 be any point, then by a *period-parallelogram* of a function, which has a primitive pair of periods 2ω, $2\omega'$, is understood the parallelogram of which the vertices are z_0, $z_0+2\omega$, $z_0+2\omega+2\omega'$ and $z_0+2\omega'$; the totality of points given by $z=z_0+2t\omega+2t'\omega'$, where $0<t<1$, $0<t'<1$, constitute the points of the parallelogram. This parallelogram is called the *period-parallelogram* z_0.

(5) A doubly-periodic function which has no essential singularity is called an *elliptic function*.

(6) By the *order* of an elliptic function is understood the number of the poles of the function inside a period-parallelogram, each pole being taken as many times as its multiplicity for this calculation.

28. The modern theory of elliptic functions may be most easily expounded by starting from the following fundamental theorems of which the first was given by Jacobi in 1834 (*Crelle's Journal*, Vol. 13) and the rest by Liouville in 1847 (see *Crelle's Journal*, Vol. 88).

(I) The ratio of the periods in any primitive pair of periods of a doubly-periodic function is necessarily non-real.

(II) An elliptic function which has no pole must be a constant.

(III) The sum of the residues of an elliptic function with respect to the poles, situated in a period-parallelogram, is zero.

(IV) An elliptic function of order 1 is impossible.

(V) An elliptic function of order n must have exactly n zeros in each period-parallelogram.

(VI) In each period-parallelogram, the sum of the values of z for which an elliptic function $f(z)$ is infinite is equal to, or differs by a period from, the sum of the values of z for which $f(z)$ is zero.

(VII) If two elliptic functions $f(z)$, $\phi(z)$, having the same periods, have, in any period-paralellogram, the same poles, and further, if in the neighbourhood of each of these poles the infinite parts of their developments are the same, then $f(z)-\phi(z)$ must be a constant.

(VIII) If two elliptic functions $f(z)$, $\phi(z)$, having the same periods, have in any period-parallelogram the same zeros and the same poles (with the same multiplicities), then $\frac{f(z)}{\phi(z)}$ must be a constant.

(IX) Between any two elliptic functions $f(z)$, $\phi(z)$, having the same periods, there exists an algebraic relation

$$G\{f(z), \phi(z)\}=0,$$

with constant co-efficients.

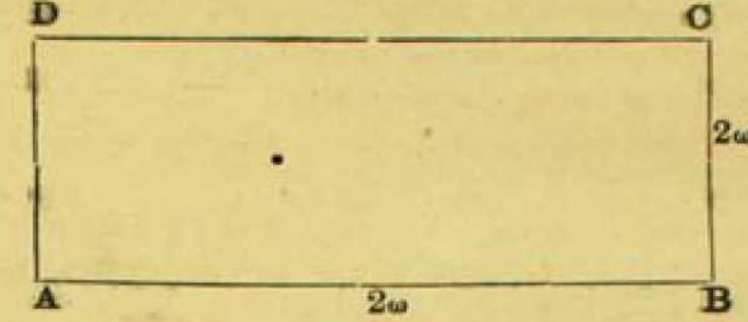
(X) Between any elliptic function $f(z)$ and its differential co-efficient $f'(z)$, there exists an algebraic relation with constant co-efficients.

(XI) Every doubly-periodic function can be expressed rationally in terms of a function of the second order, doubly-periodic with the same periods, and its differential co-efficients.

29. After the introduction of elliptic functions in Mathematics by Abel and Jacobi, Liouville's discovery of his theorems in the attempt to build up a complete theory of doubly-periodic functions starting from Jacobi's theorem (quoted above), was, according to Professor Mittag-Leffler, the first contribution of fundamental importance to the theory of elliptic functions. Liouville's first communication on the subject was made in 1844 (see *Comptes Rendus*, Vol. 19, p. 1261) but he never published a detailed account of his discovery except in the shape of a lecture which he delivered in 1847 before Borchardt and Joachimsthal and which was published by Borchardt in 1879 in *Crelle's Journal*, Vol. 88. On account of the historical importance of Liouville's theorems, I proceed to give the proofs of some of them :—

Proof of Theorem III.

Take a period-parallelogram ABCD as indicated in the adjoined figure.



Now by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{ABCD} f(z)\,dz$$

is equal to the sum of the residues of $f(z)$ inside ABCD. But

$$\int_{ABCD} f(z)\,dz=\int_{AB} f(z)\,dz+\int_{BC} f(z)\,dz+\int_{CD} f(z)\,dz+\int_{DA} f(z)\,dz.$$

Now any point z in AB has corresponding to it a point $z+2\omega'$ in CD such that $f(z)=f(z+2\omega')$; therefore

$$\int_{AB} f(z)\,dz+\int_{CD} f(z)\,dz=0,$$

the two integrals being taken in the same sense.

Similarly

$$\int_{BC} f(z)\,dz+\int_{DA} f(z)\,dz=0.$$

Therefore the sum of the residues is zero.

Hence also Theorem IV follows, for there cannot be a single pole of multiplicity unity.

Proof of Theorem V.

By a well-known theorem in the theory of functions of a complex variable,

$$\Sigma\phi(a)-\Sigma\phi(b)=\frac{1}{2\pi i}\int_{C}\phi(z)\frac{f'(z)}{f(z)}\,dz,$$

where $f(z)$ is a function having only non-essential singularities within C, $\phi(z)$ has no singularity within C, and the a's and b's are respectively the zeros and poles of $f(z)$ within C. Now apply this theorem to the case $\phi(z)=1$ and the elliptic function $f(z)$. Then $\frac{f'(z)}{f(z)}$ is itself an elliptic function. Hence by theorem III, $\int_C \frac{f'(z)}{f(z)}\,dz=0$, C being a period-parallelogram. Therefore, in C, the number of the zeros of $f(z)$ is equal to the number of the poles of $f(z)$.

Proof of Theorem VI.

In the general theorem quoted above, put $\phi(z)=z$; then taking C as a period-parallelogram, it can be easily shown that the parts of the integral due to AB and CD together give

$$-\frac{2\omega'}{2\pi i}\int_0^{2\omega}\frac{f'(z)}{f(z)}\,dz=2m_2\omega'$$

where m_2 is an integer. Similarly the parts corresponding to BC and DA give $2m_1\omega$. Thus the theorem is proved.

Proofs of Theorems VII and VIII.

In Theorem VII, since $f(z)-\phi(z)$ has no pole, by Theorem (II) it must be a constant.

Similarly in Theorem VIII, $\frac{f(z)}{\phi(z)}$ having no pole must be a constant.

30. Let us construct the simplest elliptic function with a given primitive pair of periods 2ω, $2\omega'$. In each period-parallelogram this function must have a single pole of multiplicity 2, and let one of the poles be $z=0$. Then consider the double series:

$$\text{(A)}\quad \frac{1}{z^2}+\sum_{m,n}{}'\left[\frac{1}{\{z-(2m\omega+2n\omega')\}^2}-\frac{1}{(2m\omega+2n\omega')^2}\right],$$

where Σ' indicates that the summation extends to all integral values of m, n with the exception of the combination 0, 0. Now, at the outset, it is important to note that the *double series is absolutely convergent.*

For,

$$\frac{1}{\{z-(2m\omega+2n\omega')\}^2}-\frac{1}{(2m\omega+2n\omega')^2}$$

$$=\frac{2z\left(1-\frac{z}{4m\omega+4n\omega'}\right)}{(2m\omega+2n\omega')^3\left\{1-\frac{z}{2m\omega+2n\omega'}\right\}^2},$$

which for numerically large values of m and n behaves as

$$\frac{2z}{(2m\omega+2n\omega')^3},$$

since by Jacobi's theorem $2m\omega+2n\omega'$ has an ever-increasing modulus for numerically large values of m and n. Also it is a well-known result that the double series

$$\Sigma' \frac{1}{(ma+nb)^\beta}$$

converges absolutely when $\beta > 2$, a and b being any complex quantities having a non-real ratio to each other. Thus, as in the present case β is 3, the double series is absolutely convergent; let its sum be denoted by $f'(z)$.

Now

$$f'(z) = -2\Sigma \frac{1}{\{z-(2m\omega+2n\omega')\}^3},$$

m and n having all possible integral values. Therefore it is obvious that $f'(z+2\omega)=f'(z)$, because the series for $f'(z)$ will contain the same terms when $z+2\omega$ is put for z, the order of the terms being different, and the series being absolutely convergent, change in the order of the terms will leave the sum unaltered. Therefore, integrating, $f(z+2\omega)$ equals $f(z)+$a constant. But $f(z)$ is even, since the series for $f(z)$ and $f(-z)$ are both absolutely convergent and contain the same terms arranged in different orders; and therefore $f(\omega)=f(-\omega)$. Hence the constant is zero and $f(z+2\omega)=f(z)$. Similarly $f(z+2\omega')$ equals $f(z)$. Thus $f(z)$ is an elliptic function with the given primitive pair of periods and a single pole of multiplicity 2 in each period-parallelogram.

No other elliptic function differing from $f(z)$ by other than a constant and still having the aforesaid properties, can exist. For, if possible, let another elliptic function be

$$\phi(z) = \frac{1}{z^2} + \frac{C}{z} + P(z)$$

in the neighbourhood of $z=0$, where $P(z)$ is a power series containing positive powers of z. Then by Liouville's second theorem the residue C must be zero. Therefore, applying Theorem VII, we have $f(z)-\phi(z)$ $=$constant. We will denote $f(z)$ by $\wp(z)$ when the constant term is absent from $P(z)$.

31. The above function was discovered by **Weierstrass** and is, as shown in the preceding article, the simplest elliptic function. It can be expanded in powers of z in the following manner:—

In the series (A),

$$\frac{1}{\{z-(2m\omega+2n\omega')\}^2}-\frac{1}{(2m\omega+2n\omega')^2}$$

$$=\frac{1}{(2m\omega+2n\omega')^2}\left\{1+\frac{2z}{(2m\omega+2n\omega')}+\frac{3z^2}{(2m\omega+2n\omega')^2}+\dots\right.$$

$$\left.+\dots+\frac{(r+1)z^r}{(2m\omega+2n\omega')^r}+\dots\right\}-\frac{1}{(2m\omega+2n\omega')^2}$$

Therefore

$$\wp(z)=\frac{1}{z^2}+\sum_{r=1}^{\infty}G_{r+2}\cdot(r+1)z^r,$$

where G_r stands for $\Sigma'\frac{1}{(2m\omega+2n\omega')^r}$. Now, it is clear that G_r will be zero when r is odd because m and n take all positive and negative integral values with the exception of the combination 0,0. Thus G_{r+2} is 0 unless r is even. Therefore

$$\text{(B)}\quad \wp(z)=\frac{1}{z^2}+c_2z^2+c_3z^4+\quad\dots+c_N z^{2N-2}+\dots$$

where $\quad c_N=(2N-1)\Sigma'\frac{1}{(2m\omega+2n\omega')^{2N}}.$

From (B) we have

$$\wp'(z)=-\frac{2}{z^3}+2c_2z+4c_3z^3+\dots\dots$$

Hence

$$\{\wp'(z)\}^2=\frac{4}{z^6}-\frac{8c_2}{z^2}-16c_3+\text{ positive powers of } z;$$

also

$$\{\wp(z)\}^3=\frac{1}{z^6}+\frac{3c_2}{z^2}+3c_3+\text{ positive powers of } z.$$

Therefore $\phi(z)\equiv\{\wp'(z)\}^2-4\{\wp(z)\}^3+20c_2\wp(z)=-28c_3+$ positive powers of z.

Thus the elliptic function $\phi(z)$ equals the finite quantity $-28c_3$ at $z=0$, but it cannot be infinite at any point other than $z=0$ in the period-parallelogram with $z=0$ as a vertex. Therefore $\phi(z)$ is finite everywhere in the period-parallelogram and must consequently be a constant. Therefore for every value of z, $\phi(z)=-28c_3$. Thus, denoting $20c_2$ by g_2 and $28c_3$ by g_3, we have the differential equation for $\wp(z)$:

$$\text{(C)}\qquad \{(\wp'z)\}^2=4\{\wp(z)\}^3-g_2\wp(z)-g_3.$$

32. From (C), we deduce a recurrence formula for the co-efficients c_2, c_3, etc., in the following manner :—

Differentiating (C), we have

$$2\,\wp'\wp''=12\,\wp^2\wp'-g_2\wp'$$

$$i.e.,\quad \wp''=6\wp^2-\tfrac{1}{2}\,g_2=6\wp^2-10c_2\,;$$

using in the above the expansion (B) for $\wp$, we have

$$\frac{6}{z^4}+2.1.c_2+4.3.c_3z^2+\dots(2N-2)\quad(2N-3)c_N\,z^{2N-4}+\dots$$

$$=-10c_2+6\left\{\frac{1}{z^2}+\sum_2^{\infty}c_N\,z^{2N-2}\right\}^2$$

$$=-10c_2+6\left\{\frac{1}{z^4}+2\sum_2^{\infty}c_N\,z^{2N-4}+\sum_{r,s}c_rc_s\,z^{2(r+s)-4}\right\},$$

r and s having each all the values 2, 3, 4,......to ∞. Therefore, equating the co-efficients of z^{2N-4}, we have

$$\{(2N-2)(2N-3)-12\}\,c_N=6\sum_{r+s=N}c_rc_s,\qquad N=4,5,6,\dots,$$

The above can be put in the form

$$\text{(D)}\quad (N-3)\,(2N+1)c_N=3\,[c_2c_{N-2}+c_3c_{N-3}+\dots+c_{N-2}\,c_2],$$

Calculating the c's with the help of (D) we get

$$c_4=\frac{1}{3}c_2^2,\ c_5=\frac{3}{11}\,c_2c_3,\ c_6=\frac{1}{13}\left[\,\frac{2}{3}\,c_2^3+c_3^2\,\right].$$

Thus the c's can be expressed as rational integral functions of g_2 and g_3 in the form

$$c_{\text{N}} = \Sigma a_{\lambda,\mu}\, g_2^{\lambda}\, g_3^{\mu}, \qquad \cdot \qquad \cdot$$

where the a's are numerical constants and λ, μ take all integral values subject to the condition $2\lambda + 3\mu = \text{N}$.

33. A function $\phi(z)$ is said to possess an *algebraic addition theorem*, if between $\phi(z_1+z_2)$, $\phi(z_1)$ and $\phi(z_2)$ an algebraic relation exists with co-efficients independent of z_1 and z_2. The exponential and circular functions possess such addition-theorems. I proceed to investigate such a theorem for $\wp(z)$.

Consider the function $\phi(z) \equiv \wp'(z) - a\,\wp(z) - b$, where a and b are constants to be determined so that $\phi(z_1)$ and $\phi(z_2)$ vanish. Thus, denoting $\wp(z_1)$ and $\wp(z_2)$ by p_1 and p_2 respectively, we have

$$(1) \quad ap_1 + b = p_1',$$

$$(2) \quad ap_2 + b = p_2';$$

$$i.e., \; a = \frac{p_1' - p_2'}{p_1 - p_2}, \quad b = \frac{p_1 p_2' - p_2 p_1'}{p_1 - p_2}.$$

Now $\phi(z)$ is an elliptic function which has a triple pole at $z=0$ and has no other pole in the period-parallelogram z_0. Therefore, by Liouville's Theorem (V), $\phi(z)$ has three zeros in the above parallelogram and by Liouville's Theorem (VI) the sum of the zeros of $\phi(z)$ in the above parallelogram is zero. Therefore, as z_1, z_2 are known to be two of the zeros, the other zero must be $-(z_1+z_2)$. Therefore, denoting $\wp(z_1+z_2)$ by p_3 and noting that $\wp'$ is an odd function, we have

$$(3) \quad ap_3 + b = -p'_3.$$

Thus the equation

$$\wp'^2 = 4\,\wp^3 - g_2\,\wp - g_3,$$

taken with (1), (2) and (3), shows that the algebraic equation

$$4\,t^3 - g_2 t - g_3 = (at+b)^2$$

is satisfied by $p_1 = \wp(z_1)$, $p_2 = \wp(z_2)$, $p_3 = \wp(z_1+z_2)$.

Therefore

$$(4)\quad p_1+p_2+p_3=\frac{a^2}{4}\,.$$

Hence we have the addition-theorem

$$(\mathrm{E})\quad \wp(z_1+z_2)=-\wp(z_1)-\wp(z_2)+\frac{1}{4}\left\{\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right\}^2.$$

Another form of the addition theorem can be obtained by eliminating a and b between the equations

$$p_1+p_2+p_3=\frac{a^2}{4}\,,$$

$$p_1p_2+p_2p_3+p_3p_1=-\frac{ab}{2}-\frac{g_2}{4}\,.$$

$$p_1p_2p_3=\frac{g_3}{4}+\frac{b^2}{4}\,,$$

and is

$$(p_1+p_2+p_3)\,(4p_1p_2p_3-g_3)=(p_1p_2+p_2p_3+p_3p_1+\tfrac{g_2}{4})^2.$$

FOURTH LECTURE.

THE MODERN THEORY OF ELLIPTIC FUNCTIONS—(*continued*).

34. Just as $\wp(z)$ is the simplest elliptic function, the simplest function of z, which (1) has the character of an integral function for all finite values of z, and (2) has for its zeros $z=0$ and all values of z, of the form $2m\omega+2n\omega'$, is, according to Weierstrass's factor theorem, defined by the infinite product

$$z\,\Pi'\left(1-\frac{z}{\Omega^{m,n}}\right)e^{\frac{z}{\Omega_{m,n}}+\frac{1}{2}\frac{z^2}{\Omega^2_{m,n}}}$$

where $\Omega_{m,n}$ stands for $2m\omega+2n\omega'$ and in Π' all integral values of m, n, are to be taken with the exception of the combination 0, 0. It is also understood that the real part of $\frac{\omega'}{\omega}$ is a quantity greater than zero. This function is Weierstrass's *σ-function.* It can be easily proved that

$$\sigma(0)=1,\ \sigma''(0)=0,\ \sigma'''(0)=0,\ \sigma''''(0)=0,\ \sigma^5(0)=-\frac{g_2}{2},\ \sigma^6(0)=0,$$

$$\sigma^7(0)=-6g_3.$$

Thus

$$\sigma(z)=z-\frac{g_2}{240}z^5-\frac{g_3}{840}z^7+\ldots\ldots$$

From the infinite product it is clear that $\sigma(z)$ is an odd function. Expressed in terms of $\sigma(z)$,

$$\wp(z)=-\frac{d^2}{dz^2}\{\log\sigma(z)\}=\frac{\sigma'^2(z)-\sigma(z)\sigma''(z)}{\sigma^2(z)}.$$

The function $\frac{d}{dz}\{\log\sigma(z)\}$ is called the *ζ-function.*

The expanded form of ζ is

$$\frac{1}{z}+\Sigma'\left(\frac{1}{z-\Omega_{m,n}}+\frac{1}{\Omega_{m,n}}+\frac{z}{\Omega^2_{m,n}}\right)$$

It should be noted that neither $\sigma(z)$ nor $\zeta(z)$ is a periodic function. In fact

$$\sigma(z+\Omega_{m,n})=\epsilon\sigma(z)\times e^{\eta\left(z+\frac{\Omega_{m,n}}{2}\right)}$$

and
$$\zeta(z+\Omega_{m,n})=\zeta(z)+\eta,$$

where $\eta=m\eta_1+n\eta_2$, $\epsilon=+1$ or -1 according as $\frac{1}{2}\Omega_{m,n}$ is a period or not,

$$\eta_1=\zeta(z+2\omega)-\zeta(z),\ \eta_2=\zeta(z+2\omega')-\zeta(z),$$

i.e.,
$$\eta_1=2\zeta(\omega),\ \eta_2=2\zeta(\omega').$$

35. Although σ and ζ are not periodic functions, each can be used to express any elliptic function, and the following theorems, (F), (G) hold.

(F) Every elliptic function $f(z)$ admits of being expressed in the form

$$f(z)=\frac{C\sigma(z-b_1)\sigma(z-b_2)\ldots\sigma(z-b_r)}{\sigma(z-a_1)\sigma(z-a_2)\ldots\sigma(z-a_r)},$$

where C is a constant, the b's form a complete system of the zeros, and the a's form a complete system of the poles, of the function $f(z)$, and they are so chosen that

$$b_1+b_2+\ldots+b_r=a_1+a_2+\ldots+a_r\,;$$

thus if the zeros β and the poles α of $f(z)$ are assigned beforehand, we have to find points a, b homologous with α and β so that, although

$$\Sigma\alpha-\Sigma\beta=\text{a period},$$

$$\Sigma a-\Sigma b=0.$$

Examples. (1) $$\wp'(z)=\frac{2\sigma(z-\omega)\sigma(z-\omega')\sigma(z-\omega'')}{\sigma(\omega)\sigma(\omega')\sigma(\omega'')\sigma^3(z)},$$

where $-\omega''=\omega+\omega'$.

(2) $$\wp(z_1)-\wp(z_2)=-\frac{\sigma(z_1+z_2)\sigma(z_1-z_2)}{\sigma^2(z_1)\sigma^2(z_2)}.$$

(3) $$\wp'(z)=-\frac{\sigma(2z)}{\sigma^4(z)}.$$

(G) Every elliptic function $f(z)$ admits of being expressed in terms of ζ and its differential coefficients in the form

$$f(z)=C+\sum_a \Big\{ A\zeta(z-a)+A'\zeta'(z-a)+A''\zeta''(z-a) + \ldots + A^{(k-1)}\,\zeta^{(k-1)}(z-a)\Big\},$$

where the summation extends over the different poles (in a period-parallelogram) of the function ; the constants A, A′, etc., correspond to the fractional part

$$\frac{A}{z-a}-\frac{A'}{(z-a)^2}+\frac{2\,!\,A''}{(z-a)^3}-\ldots+\frac{(-1)^{k-1}\;(k-1)!A^{(k-1)}}{(z-a)^k}$$

of the function in the neighbourhood of a, and C is a constant.

36. I proceed now to consider the expression of any elliptic function $f(z)$ by means of the $\wp$-function and its differential coefficient.

First, suppose that $f(z)$ is an even function. Let those of its zeros which are not periods be

$$\pm b_1+\text{period},\ \pm b_2+\text{period},\ldots$$

and let μ_1, μ_2, etc., be the respective orders of multiplicity. Also let $\pm a_1+$period, $\pm a_2+$period,...be those of the poles which are not periods, and let the respective orders of multiplicity be λ_1, λ_2, etc. Then $f(z)$ must be equal to

$$\text{(H)}\qquad C\frac{\{\wp(z)-\wp(b_1)\}^{\mu_1}\ \{\wp(z)-\wp(b_2)\}^{\mu_2}\ \ldots}{\{\wp(z)-\wp(a_1)\}^{\lambda_1}\ \{\wp(z)-\wp a_2)\}^{\lambda_2}\ \ldots},$$

where C is a constant. For, denoting the above expression by $\phi(z)$, we have $f(z)/\phi(z)$, an elliptic function which, outside the periods, admits of neither a zero nor a pole. But a period cannot be both a zero and a pole. Therefore $f(z)/\phi'z)$ lacks zeros or poles, and it must, therefore, be a constant. By giving a particular value to z, one can choose C so that the aforesaid constant is unity.

Secondly, let $f(z)$ be not necessarily even. Then $\dfrac{f(z)+f(-z)}{2}$ is even, and $\dfrac{f(z)-f(-z)}{2\wp'(z)}$ is also even; and, consequently, each is expressible in terms of the $\wp$-function as in (H). Therefore

$$f(z)=\mathrm{R}\{\wp(z)\}+\wp'(z)\mathrm{R}_1\{\wp(z)\},$$

where R and R_1 are both rational functions.

It should be noted that, as $\wp(\mathrm{N}z)$ is an even function, it is expressible as a rational function of $\wp(z)$, N being any integer.

Ex. $\wp(2z)=\dfrac{\wp^4+\frac{1}{2}g_2\wp^2+2g_3\wp+\frac{1}{16}g_2^2}{4\wp^3-g_2\wp-g_3}$.

37. The expression for $\wp(\mathrm{N}z)$ in terms of $\wp(z)$ can be most easily obtained as follows :—

The poles of $\wp(\mathrm{N}z)$ are the origin and the (N^2-1) points $\dfrac{\Omega_{m,n}}{\mathrm{N}}$, m and n having all possible integral values from 0 to $\mathrm{N}-1$, the combination 0,0 being excluded. If, therefore, we put $z=h$ or $\dfrac{\Omega_{m,n}}{\mathrm{N}}+h$, where h is very small,

$$\wp(\mathrm{N}z)=\wp(\mathrm{N}h)=\frac{1}{\mathrm{N}^2h^2}+c_2\mathrm{N}^2h^2+\dots$$

But, using the above, the theorem (G) gives

$$\wp(\mathrm{N}z)=-\frac{1}{\mathrm{N}^2}\,\zeta'(z)-\frac{1}{\mathrm{N}^2}\,\Sigma\zeta'\Big(z-\frac{\Omega_{m,n}}{\mathrm{N}}\Big)+\text{const.}$$

$$\textit{i.e.}\quad \mathrm{N}^2\wp(\mathrm{N}z)=\wp(z)+\Sigma\wp\Big(z-\frac{\Omega_{m,n}}{\mathrm{N}}\Big)+\mathrm{C}\qquad\dots\qquad(1)$$

Consider the neighbourhood of the origin ; then the above gives

$$\mathrm{C}=-\Sigma\wp\Big(-\frac{\Omega_{m,n}}{\mathrm{N}}\Big)=-\Sigma\wp\Big(\frac{\Omega_{m,n}}{\mathrm{N}}\Big)\qquad\dots\qquad(2)$$

Integrating (1) we have

$$N\zeta(Nz)=\zeta(z)+\sum\zeta\left(z-\frac{\Omega_{m,n}}{N}\right)-Cz+C', \quad \ldots \quad 3)$$

whence, considering the neighbourhood of the origin, we have

$$C'=\sum\zeta\left(\frac{\Omega_{m,n}}{N}\right).$$

Now change z into $z+2\omega$ in (3), then by the result given at the end of Art. 34,

$$N\zeta(Nz)+N^2\eta_1=\zeta(z)+\eta_1+\sum\left\{\zeta\left(z-\frac{\Omega_{m,n}}{N}\right)+\eta_1\right\}$$

$$-C(z+2\omega)+C$$

$$=\zeta(z)+\sum\zeta\left(z-\frac{\Omega_{m,n}}{N}\right)+N^2\eta_1-C(z+2\omega)+C'.$$

Therefore, expanding both the sides in powers of z and equating the coefficients of z, C must be zero, *i.e.*, $\sum\wp\left(\frac{\Omega_{m,n}}{N}\right)=0$.

Thus $$N^2\wp(Nz)=\wp(z)+\sum\wp\left(z-\frac{\Omega_{m,n}}{N}\right) \quad \ldots \quad (I)$$

38. Applying the addition theorem, the above becomes

$$N^2\wp(Nz)=\wp(z)+\sum\left[\frac{1}{4}\left\{\frac{\wp'(z)-\wp'\left(\frac{\Omega_{m,n}}{N}\right)}{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)}\right\}^2\right.$$

$$\left.-\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)\right].$$

As $\sum\wp\left(\frac{\Omega_{m,n}}{N}\right)=0$ and as obviously the odd part in the above must be absent, the above becomes

$$N^2\wp(Nz)=\wp(z)+\Sigma\left[\frac{1}{4}\,\frac{\{\wp'(z)\}^2+\left\{\wp'\left(\frac{\Omega_{m,n}}{N}\right)\right\}^2}{\left\{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)\right\}^2}-\wp(z)\right] \qquad (1)$$

Now it is easily seen that

$$\wp'^2(z)+\wp'^2\left(\frac{\Omega_{m,n}}{N}\right)=4\left\{\wp^2(z)+\wp(z)\wp\left(\frac{\Omega_{m,n}}{N}\right)+\wp^2\left(\frac{\Omega_{m,n}}{N}\right)-\tfrac{1}{4}g_2\right\}$$

$$\times\left\{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)\right\}+2\wp'^2\left(\frac{\Omega_{m,n}}{N}\right)$$

and

$$\wp^2(z)+\wp(z)\wp\left(\frac{\Omega_{m,n}}{N}\right)+\wp^2\left(\frac{\Omega_{m,n}}{N}\right)=\left\{\wp(z)+2\wp\left(\frac{\Omega_{m,n}}{N}\right)\right\}$$

$$\times\left\{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right\}+3\wp^2\left(\frac{\Omega_{m,m}}{N}\right).$$

Therefore (1) becomes

$$N^2\wp(Nz)=\wp(z)+\Sigma\frac{3\wp^2\left(\frac{\Omega_{m,n}}{N}\right)-\frac{1}{4}g_2}{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)}+\Sigma\frac{\frac{1}{2}\left\{\wp'\left(\frac{\Omega_{m,n}}{N}\right)\right\}^2}{\left\{\wp(z)-\wp\left(\frac{\Omega_{m,n}}{N}\right)\right\}^2}.$$

The multiplication theorems for ζ and σ functions are respectively

$$N\zeta(Nz)=\zeta(z)+\Sigma\zeta\left(z-\frac{\Omega_{m,n}}{N}\right)-N(N-1)\eta_s$$

and

$$\sigma(Nz)=N\sigma(z)\ \Pi\frac{\sigma\left(z-\frac{\Omega_{m,n}}{N}\right)}{\sigma\left(-\frac{\Omega_{m,n}}{N}\right)}\times e^{-N(N-1)\eta_s z},$$

where η_s is a constant.

Ex. $\wp(3z)=\wp(z)+\{-2\wp^6+\frac{5}{2}g_2\wp^4+10g_3\wp^3+\frac{5}{8}g_2^2\wp^2+\frac{1}{2}g_2g_3\wp$

$+g_3^2+\frac{1}{32}g_2^4\}$

$\div\{3\wp^4-\frac{3}{2}g_2\wp^2-3g_3\wp-\frac{1}{6}g_2^2\}^2$

39. Abel's problem of *complex multiplication* with reference to the $\wp$-function may be thus enunciated.[1] What particular conditions must be satisfied by the function and the number m in order that $\wp(mz)$ may be expressible as a rational function of $\wp(z)$? I proceed to answer the above question.

If 2ω, $2\omega'$ be a primitive pair of periods of $\wp(mz)$, then, according to a well-known theorem,

$$2\omega=\frac{2a\omega}{m}+\frac{2b\omega'}{m},\ 2\omega'=\frac{2c\omega'}{m}+\frac{2d\omega'}{m},$$ a, b, c and d being all integers. Thus

$$m\omega=a\omega+b\omega',\ m\omega'=c\omega+d\omega' \quad \ldots \qquad (1)$$

which equations are satisfied if $b=0$, $c=0$, $a=d$, m being integral. If m is not integral, then eliminating m between the equations (1) we have

$$b\omega'^2+(a-d)\,\omega'\omega-c\,\omega^2=0.$$

Thus if m is not integral the necessary conditions for the representability of $\wp(mz)$ as a rational function of $\wp(z)$ are (1) that the ratio $\frac{\omega'}{\omega}$ must be a root of a quadratic equation with integral coefficients and

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[1] In his memoir, "Recherches sur les fonctions elliptiques" (*Crelle's Journal*, Vols. 2 and 3, 1827, 1828), Abel introduced the notion of the complex multiplication of elliptic functions. Denoting the integral $\int_0^x \frac{dx}{\sqrt{1-x^4}}$ by δ and the lemniscate function by $x=\phi(\delta)$, Abel gave the first example (*Oeuvres*, t. 1, p. 354) of complex multiplication, *viz.*,

$$\phi\{(2+i)\delta\}=xi\frac{1-2i-x^4}{1-(1-2i)x^4}.$$

He worked out for the sn-function the cases of the multiplicators $\sqrt{-3}$, $\sqrt{-5}$ (*Oeuvres*, t. 1, pp. 377-382). The subject of complex multiplication was developed subsequently, chiefly by Kronecker (see *Monatsberichte der Akademie der Wissenschaften zu Berlin*, for 1857 and 1862), Hermite (*Oeuvres*, t. 2), H. Weber (*Lehrbuch der Algebra*, Bd. 3, 1908) and R. Fueter (*Vorlesungen ueber die singulären Moduln und die Komplexe Multiplikation der elliptischen Funktionen*, 1924).

Among English writers on the subject may be mentioned Cayley, Greenhill and Russell, who published a number of papers in the *Proceedings of the London Mathematical Society*, Series 1.

having imaginary roots, and (2) that m must be of the form $a+b\tau$, where τ is that root of the aforesaid equation which has its imaginary part positive, a and b being integers. It can be proved without difficulty that, if it is known that the condition (1) is satisfied, an infinite number of complex multiplicators m can be found which must be of the form

$$y+ix\sqrt{D} \text{ or } \frac{y'+ix'\sqrt{D}}{2},$$

where $D=AC-B^2$, the known equation for τ being

$$A\tau^2+2B\tau+C=0,$$

and x, y, x', y' are integers.

Ex. Let $\tau^2+1=0$, then $D=1$ and $m=y+ix$; x and y being any integers. For instance $\wp(iz)=-\wp(z)$.

Ex. Let $2\tau^2+2\tau+2=0$; then $D=3$ and $m=\dfrac{y'+ix'\sqrt{3}}{2}$, x', y' being any integers. For instance $\wp(\omega z)=\omega\wp(z)$, ω being a cube root of unity.

40. The proof of the statement made in the end of the above article is as follows :—

As the roots are imaginary, $AC-B^2\equiv D$ is positive. Therefore, without any loss of generality, we may assume that A, C are positive and that A, B, C are prime to one another.

First, let A, 2B, C have for their greatest common measure unity.

Now, for complex multiplication, it is necessary that

$$A\tau^2+2B\tau+C=0$$

should be equivalent to

$$b\tau^2+(a-d)\tau-c=0.$$

Therefore we must have

$$b=Ax,\ (a-d)=2Bx,\ c=-Cx.$$

Let $2y$ denote $a+d$, then we have

$$a=y+Bx,\ b=Ax,\ c=-Cx,\ d=y-Bx.$$

Also the *determinant* $n\equiv ad-bc=y^2+Dx^2$; and

$$\tau=\frac{-B+i\sqrt{D}}{A},$$

$$m=a+b\tau=y+ix\sqrt{D}.$$

Since A, 2B, C are prime to one another ; in order that a, b, c, d may be integers, x and y must be also integers.

Secondly, let A, 2B, C have for their greatest common measure 2. Then x and y may be halves of integers x', y'.

Thus

$$a=\frac{y'+Bx'}{2},\ b=\frac{Ax'}{2},\ c=-\frac{Cx'}{2},\ d=\frac{y'-Bx'}{2}.$$

Also the determinant $ad-bc=\dfrac{y'^2+Dx'^2}{4}$; and

$$m=\frac{y'+ix'\sqrt{D}}{2}$$

In this case, A and C are even, B is odd and D is of the form $4N-1$, N being an integer.

41. Let $\mathfrak{E}(z)$ and $\bar{\mathfrak{E}}(z)$ be two elliptic functions, the periods of which are connected by the relations

$$\omega_1=a\bar{\omega}_1+b\bar{\omega}_2,\ \omega_2=c\bar{\omega}_1+d\bar{\omega}_2.$$

where $ad-bc=n$.

Their invariants J and $\bar{J}$ are connected by a modular equation

$$F_n(J, \bar{J})=0,$$

and the different values of $\bar{J}$ furnished by this equation correspond to the different ways of determining a, b, c, d under the condition that $ad-bc$ should be n.

If $\bar{J}$ is to be equal to J, then it is necessary and sufficient that we should have

$$\frac{\bar{\omega}_2}{\bar{\omega}_1}=\frac{\omega_2}{\omega_1};$$

whence

$$\bar{\omega}_1 = \frac{\omega_1}{m}, \quad \bar{\omega}_2 = \frac{\omega_2}{m},$$

m denoting a constant. We have, therefore,

$$m\,\omega_1 = a\,\omega_1 + b\omega_2, \quad m\omega_2 = c\omega_1 + d\omega_2\,;$$

and the function $\wp(z)$ admits of complex multiplication.

The equation

$$F_n\,(J, J) = 0$$

resolves into rational factors, giving respectively the invariants **J** which correspond to the two types of complex multiplication, *viz*.,

(1) those in which

$$n = x^2 + Dy^2,$$

x, y being integers prime to each other;

(2) those in which

$$4n = x'^2 + Dy'^2,$$

x', y' being odd and prime to each other.

42. We know that $\wp\,(Nz) = \dfrac{R\{\wp(z)\}}{R_1\{\wp(z)\}}$

where R and R_1 are both rational integral functions.

Thus, putting z for Nz, we have

$$\wp(z) = \frac{R\left\{\wp\left(\frac{z}{N}\right)\right\}}{R_1\left\{\wp\left(\frac{z}{N}\right)\right\}}$$

or

$$\frac{R(x)}{R_1(x)} = y,$$

where y stands for $\wp(z)$ and x for $\wp\left(\frac{z}{N}\right)$. It is obvious that the roots of the above algebraic equation in x are all given by

$$\text{(1)} \qquad x = \wp\left(\frac{\pm z + \Omega_{m,n}}{N}\right),$$

m and n being given all integral values, positive and negative.

(*a*) If N is odd, all the possible values which (1) can take are obtained from

$$\wp\left(\frac{z+\Omega_{m,n}}{N}\right),$$

m and n taking all the integral values from $-\frac{N-1}{2}$ to $\frac{N+1}{2}$, including the extreme values. All these values of

$$\wp\left(\frac{z+\Omega_{m,n}}{N}\right)$$

are different and they are N^2 in number.

(*b*) If N is even, all the possible values of (1) are obtained from

$$\wp\left(\frac{z+\Omega_{m,n}}{N}\right),$$

m and n being given all the integral values from $-\frac{N-2}{2}$ to $\frac{N}{2}$, both the extreme values being included. These values of

$$\wp\left(\frac{z+\Omega_{m,n}}{N}\right)$$

are all different and are N^2 in number.

Thus the equation $R_1(z).y-R(z)=0$ has N^2 different roots. They are all simple roots. For, otherwise, if z were a multiple root, then $R'_1(z)y-R'(z)=0$. Thus $RR'_1-R_1R'=0$,

$$i.e., \ \frac{d}{dz}\left(\frac{R}{R_1}\right)=0$$

independently of z.

Hence $\frac{R}{R_1}$ will be independent of z and consequently y will be a constant, independent of z, which is not the case.

43. Let $-2D$ stand for the operator

$$12g_3\frac{\partial}{\partial g_2}+\frac{2}{3}g_2^2\ \frac{\partial}{\partial g_3};$$

then the following partial differential equations hold :—

$$D\{\wp(z)\}=-\zeta(z)\cdot\wp'(z)-2\wp^2(z)+\tfrac{1}{3}g_2. \qquad \ldots \quad (1)$$

$$D\{\zeta(z)\}=\zeta(z)\wp(z)+\tfrac{1}{2}\wp'(z)-\tfrac{1}{12}g_2 z. \qquad \ldots \quad (2)$$

$$D\{\log\sigma(z)\}=-\tfrac{1}{2}\zeta^2(z)+\tfrac{1}{2}\wp(z)-\tfrac{1}{24}g_2 z^2. \qquad \ldots \quad (3)$$

The last equation gives

$$\frac{\partial^2\sigma(z)}{\partial z^2}-12g_3\frac{\partial\sigma(z)}{\partial g_2}-\frac{2}{3}g_2^2\frac{\partial\sigma(z)}{\partial g_3}+\tfrac{1}{12}g_2z^2\sigma(z)=0 \quad \text{(K)}$$

The above equation was used by Weierstrass to obtain the coefficients in the expansion of $\sigma(z)$ in powers of z.

If we put

$$\sigma(z)=\sum_{m,n}a_{m,n}\,(\tfrac{1}{2}g_2)^m(2g_3)^n\cdot\frac{z^{4m+6n+1}}{(4m+6n+1)!},$$
$$(m,n=0,1,2\,\ldots\ldots\infty)$$

then the following recurrence formula is obtained from (K) for the calculation of the a's :

$$a_{m,n}=3(m+1)a_{m+1,n-1}+\tfrac{16}{3}(n+1)a_{m-2,n+1}$$
$$-\tfrac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}.$$

To the coefficient $a_{0,0}$ the value 1 is to be attributed, and to any coefficient of which one of the two subscripts is negative the value 0 is to be given. Weierstrass obtained the expansion of σ up to the 35th power of z.

44. I will conclude my exposition of the modern theory by proving the proposition that every elliptic function $\phi(z)$ has an algebraic addition-theorem and by stating the converse of the proposition.

Proof :

It has been shown in Art. 36 that every elliptic function is expressible rationally in terms of $\wp$ and $\wp'$. Therefore

$$\phi(z_1+z_2)=\mathrm{R}\{\wp(z_1+z_2),\ \wp'(z_1+z_2)\}. \quad \ldots \quad (1)$$

Putting for brevity

$$\wp(z_1)=p_1,\ \wp'(z_1)=p_1',\ \wp(z_2)=p_2,\ \wp'(z_2)=p_2',$$

we have, according to the addition-theorem (E),

$$\wp(z_1+z_2)=\mathrm{R}_1(p_1,\ p_1',\ p_2,\ p_2') \quad \ldots \quad (2)$$

where R_1 denotes a rational function with coefficients independent of z_1 and z_2.

Differentiating (2) with respect to z_1, we have

$$\wp'(z_1+z_2)=\frac{\partial R_1}{\partial p_1}\cdot p_1'+\frac{\partial R_1}{\partial p'_1}\cdot\wp''(z_1)=R_2(p_1, p'_1, p_2, p_2'), \qquad (3)$$

where for $\wp''(z_1)$ is substituted $6p_1^2-\frac{1}{2}g_2$ and R_2 is a rational function with coefficients independent of z_1 and z_2.

Substituting the expression (2) and (3) in (1), we have

$$\phi(z_1+z_2)=R_3(p_1, p_2, p_1', p_2'). \qquad \ldots \quad (4)$$

If we combine this equation with the four equations

$$\phi(z_1)=R(p_1, p_1'), \quad \phi(z_2)=R(p_2, p_2'),$$

$$p_1'^2=4p_1^3-g_2p_1-g_3, \quad p_2'^2=4p_2^3-g_2p_2-g_3,$$

and eliminate the four quantities p_1, p_1', p_2', p_2, we obtain an algebraic equation of the form

$$G\{\phi(z_1+z_2), \phi(z_1), \phi(z_2)\}=0$$

in which the coefficients are independent of z_1 and z_2.

The converse of the above proposition is the following: Every function for which there exists an algebraic addition-theorem must be an elliptic function or a limiting case of one, that limiting case being the rational, the trigonometric or the exponential function respectively, according as both ω and ω' are infinite, ω' is infinite, or ω is infinite.

FIFTH LECTURE

APPLICATIONS OF ELLIPTIC FUNCTIONS

45. One of the earliest applications of elliptic functions was found in the exact solution of the problem of the simple pendulum. Starting with the well-known equation of motion

$$l\frac{d^2\theta}{dt^2}=-g\sin\theta,$$

we have as its first integral

$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}.\sqrt{\cos\theta-\cos\alpha},$$

which becomes

$$d\left(t\sqrt{\frac{g}{l}}\right)=\frac{dx}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2 x}},$$

where x is given by

$$\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin x.$$

Thus

$$t\sqrt{\frac{g}{l}}=\int_0^x\frac{dx}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2 x}}$$

which, according to the definition of the sn-function, leads to

$$\sin x=\operatorname{sn}\left(t\sqrt{\frac{g}{l}},k\right)$$

where k stands for $\sin\frac{\alpha}{2}$.

Therefore $\sin\frac{\theta}{2}=k\operatorname{sn}\left(t\sqrt{\frac{g}{l}},k\right)$.

and half the period, T, is given in terms of 2K by

$$T\sqrt{\frac{g}{l}}=2K,$$

i.e.,

$$T=2K\sqrt{\frac{l}{g}},$$

K standing for

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\sin^2\frac{a}{2}\sin^2 x}}.$$

46. The solution of the problem of the spherical pendulum necessitates the use of the elliptic functions. This fact was known to Lagrange and the first treatment of the problem with the help of elliptic functions was given by Richelot in *Crelle's Journal* in 1852.

Taking the axis of z vertically upwards, we have, from the well-known two integrals of the equation of motion, giving respectively the area-theorem and the energy-theorem,

$$a^2\left(\frac{dz}{dt}\right)^2=2g(z_0-z)(a^2-z^2)-h^2.$$

By Descartes' Rule of signs, it is clear that the right-hand side when equated to zero gives real roots; in fact two of them, z_1 and z_2, satisfy the inequality

$$-a\leq z_1\leq z_2\leq a.$$

Measuring t from the moment when z has the minimum value z_1, we have

$$z=\frac{z_0}{3}+\frac{2a^2}{g}\wp\left(t+\frac{\omega_1}{2}\right),$$

where ω_1 is the imaginary, and ω_2 the real, period of the $\wp$-function. The maximum is attained when $t=\frac{\omega_2}{2}$. The angle $\tan^{-1}\frac{y}{x}\equiv\phi$ can be expressed in terms of Weierstrass's σ_1-function by the equation.

$$\phi=\frac{1}{2i}\log\left\{\frac{\sigma_1(t+u_1)\sigma_1(t+u_2)}{\sigma_1(t-u_1)\sigma_1(t-u_2)}\right\}+it[\zeta(u_1)+\zeta(u_2)],$$

ϕ being measured from its position when $t=0$ and u_1, u_2 are two constants.

47. The motion of a rigid body under no forces was discussed by Legendre in his "Traite des fonctions elliptiques," Vol. I, with the use of elliptic integrals. The equations of motion of Euler are

$$A\frac{dp}{dt}=(B-C)qr,$$

$$B\frac{dq}{dt}=(C-A)pr,$$

$$C\frac{dr}{dt}=(A-B)pq,$$

where A, B, C, are the principal moments of inertia and p, q, r, are the angular velocities with reference to the principal axes fixed in the body. The equations have obviously

$$Ap^2+Bq^2+Cr^2=D\mu^2,$$

$$A^2p^2+B^2q^2+C^2r^2=D^2\mu^2$$

as two integrals, D and μ being constants. Eliminating q and r between the first equation of Euler and the above two integrals, we have

$$a\frac{dp}{dt}=\sqrt{(1-\theta^2p^2)(1-p^2)},$$

where a and θ are constants connected with A, B, C, D, μ. Thus p is an elliptic function of t; similarly q and r are elliptic functions of t. In fact

$$p=\text{constant}\times\text{cn}(\lambda t),$$

$$q=\text{constant}\times\text{sn}(\lambda t),$$

$$r=\text{constant}\times\text{dn}(\lambda t),$$

where λ as well as the modulus is expressible easily in terms of A, B, C, D, μ.

48. Elliptic functions occur in the solution of a number of famous problems of conformal representation; of these some may be enumerated here :—(*a*) The inside of the circle in the s-plane with $s=0$ as its centre

and radius unity is conformally representated on the outside of a square in the u-plane by the formula

$$u=\int_{1}^{s}\frac{\sqrt{1+s^4}}{s^2}ds,$$

and thereby is solved the problem of the determination of the stationary state of heat, of the inside plane bounded by the square, with prescribed boundary conditions.

(*b*) The inside of an ellipse in the u-plane with the foci $u=\pm1$, and vertices $u=\pm a\pm ib$ is conformally represented on the inside of a circle in the s-plane with the centre $s=0$ and radius $\frac{1}{\sqrt{k}}$ by the formula,

$$s=\text{sn}\left(\frac{2K}{\pi}\sin^{-1}u\right),$$

if the modulus k is determined by the equations

$$q=\left(\frac{a-b}{a+b}\right)^2,$$

$$k=4q^{\frac{1}{2}}\left\{\frac{(1+q^2)(1+q^4)\ldots\ldots}{(1+q)(1+q^3)\ldots\ldots\ldots}\right\}^4$$

(*c*) By means of elliptic integrals of the third kind, which contain the elliptic co-ordinates of a point on the ellipsoid, the surface of an ellipsoid of three unequal axes can be conformally represented on a plane so that one half of the ellipsoidal surface, separated by the ellipse going through its four circular points, corresponds to the inside of a rectangle the vertices of which correspond to the four circular points.

49. Some problems in electrostatics have been easily solved with the help of elliptic functions. For example, taking ϕ to be the potential and ψ the stream function,

$$\text{sn}\,(\phi+i\psi)=e^{\frac{x+iy}{b}}$$

gives the electrical distribution over a pile of parallel strips of width, x_2-x_1, the distance between the consecutive strips being πb, alternate strips being at the same potential. The potential of one set of plates is K, that of the other is $-$K.

Similarly we can deal with the case of a system of $2n$ plates arranged radially and making equal angles with each other, the alternate plates being at the same potential, and the extremities of the plates lying on two co-axial right circular cylinders.

50. The problem of the conduction of heat in a solid ellipsoid of three unequal axes a, b, c can be elegantly treated with the help of elliptic functions. Let λ, μ, ν be the parameters of the three confocal conicoids through any point (x, y, z). Also, denote by e_1, e_2, e_3 respectively,

$$\frac{1}{3}(b^2+c^2-2a^2),\quad \frac{1}{3}(c^2+a^2-2b^2),\quad \frac{1}{3}(a^2+b^2-2c^2),$$

and introduce the co-ordinates u, v, w so that

$$\wp(u)=-\lambda+\frac{1}{3}(a^2+b^2+c^2),$$

$$\wp(v)=-\mu+\frac{1}{3}(a^2+b^2+c\ ),$$

$$\wp(w)=-\nu+\frac{1}{3}(a^2+b^2+c^2),$$

Then the operator ∇^2 is equivalent to

$$\frac{1}{\{\wp(u)-\wp(v)\}\{\wp(u)-\wp(w)\}}\frac{\partial^2}{\partial u^2}+\frac{1}{\{\wp(v)-\wp(u)\}\{\wp(v)-\wp(w)\}}\frac{\partial^2}{\partial v^2}$$
$$+\frac{1}{\{\wp(w)-\wp(u)\}\{\wp(w)-\wp(v)\}}\frac{\partial^2}{\partial w^2}.$$

Therefore the equation of the stationary state of heat is

$$\{\wp(v)-\wp(w)\}\frac{\partial^2 V}{\partial u^2}+\{\wp(w)-\wp(u)\}\frac{\partial^2 V}{\partial v^2}+\{\wp(u)-\wp(v)\}\frac{\partial^2 V}{\partial w^2}=0;$$

and has for a solution.

$$V=F(u)\ F\ v)\ F(w)$$

where $F(s)$ satisfies the equation of Lame, *viz.*,

$$\frac{d^2F}{ds^2}=\{A\wp(s)+B\}F,$$

A and B being constants.

Similarly the equation of the non-stationary state of heat, *viz.*,

$$\nabla^2 V = \frac{1}{K^2}\frac{\partial V}{\partial t}$$

has for a solution $V = F(u)\,F(v)\,F(w)e^{AK^2t}$

where A is any negative arbitrary constant and $F(s)$ satisfies the differential equation

$$\frac{d^2F}{ds^2} = \{A(\wp s)^2 + B\wp s + C\}F,$$

B and C being constants.

51. I proceed now to explain some applications of elliptic functions to Pure Mathematics, taking up first the theory of equations of the fifth degree, discussing it at some length and contenting myself later with brief expositions of the applications to curves of the third and higher degrees. I will conclude with a treatment of the problem of the polygon of Poncelet.

As is well-known, Abel proved in 1826 that the roots of an equation of the fifth degree cannot in general be derived from its co-efficients by using radicals alone. But in the year 1858 both Hermite and Kronecker found the solution of the general quintic by the use of elliptic functions. I propose to state clearly what Hermite did, leaving aside the treatment of Kronecker for want of time.

(*i*) The general equation of the fifth degree can be reduced to the form

$$x^5 - x - A = 0,$$

by a substitution of which the co-efficients are determined by the use of no other irrationalities than square and cube roots. This result is due to the Swedish mathematician, Bring, who gave it in 1786. Now, just as the general cubic, reduced to the form

$$x^3 - 3x + 2a = 0,$$

can be solved by representing the co-efficient a as sin-α, the roots being

$$2\sin\frac{\alpha}{3},\ 2\sin\frac{\alpha+2\pi}{3},\ 2\sin\frac{\alpha+4\pi}{3},$$

the equation $x^5 - x - A = 0$ can be solved by expressing A in terms of elliptic functions.

(*ii*) If k be the modulus of the elliptic integral

$$\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

and l the modulus of the integral

$$\text{M.}\int\frac{dy}{\sqrt{(1-y^2)(1-l^2y^2)}},$$

which results from a transformation of the nth order

$$y=\frac{a_0+a_1x+a_2x^2+\;.\;.\;.\;.+a_nx^n}{b_0+b_1x+\;.\;.\;.\;.+b_nx^n},$$

where n is an odd prime; then $u=k^{\frac{1}{4}}$ and $v=l^{\frac{1}{4}}$ are connected by an equation of the $(n+1)$th degree called the modular equation. For $n=5$, the equation is, as stated in the first lecture,

$$u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0.$$

If u be expressed in terms of $q\equiv e^{-\pi K'/K}$, *e.g.*, as follows:

$$u=\sqrt{2}q^{\frac{1}{8}}\,\frac{\Sigma q^{2m^2+m}}{\Sigma q^{m^2}}\,[=\phi\,(\omega)],\quad (m=-\infty\ldots+\infty)$$

we obtain the $(n+1)$ values of v, which satisfy the modular equation, on inserting, in the place of $q^{\frac{1}{8}}$ in the above formula, in order

$$q^{\frac{n}{8}},\; q^{\frac{1}{8n}},\; \alpha q^{\frac{1}{8n}},\;\ldots\ldots,\; \alpha^{n-1}\,q^{\frac{1}{8n}},$$

where $\alpha=e^{\frac{2\pi i}{n}}$. Thus, knowing u, we can obtain q and thereby the values of v.

(*iii*) If $q=e^{i\pi\omega}$ and $u=\phi(\omega)$, then the six roots of the equation in v are

$$-\phi(5\omega),\;\phi\Big(\frac{\omega}{5}\Big),\;\phi\Big(\frac{\omega+16}{5}\Big),\;\phi\Big(\frac{\omega+2.16}{5}\Big),\;\phi\Big(\frac{\omega+3.16}{5}\Big),$$

$$\phi\Big(\frac{\omega+4.16}{5}\Big).$$

Now construct the equation whose roots are

$$\Phi(\omega),\ \Phi(\omega+16),\ \Phi(\omega+2.16),\ \Phi(\omega+3.16),\ \Phi(\omega+4.16),$$

where $\Phi(\omega)$ stands for

$$\left[\phi(5\omega+\phi\left(\frac{\omega}{5}\right)\right]\left[\phi\left(\frac{\omega+16}{5}\right)-\phi\left(\frac{\omega+4.16}{5}\right)\right]$$

$$\times\left[\phi\left(\frac{\omega+2.16}{5}\right)-\phi\left(\frac{\omega+3.16}{5}\right)\right].$$

Then the equation will be

$$y^5-2^4.\,5^3u^4(1-u^8)^2y-2^6\sqrt{5^5}.\,u^3(1-u^8)^2(1+u^8)=0.$$

Comparing this with the Bring form of the general quintic,

$$x^5-x-\mathrm{A}=0,$$

we have, on putting

$$y=2\sqrt[4]{5^3}.\,u.\sqrt{1-u^8}\quad x,$$

$$\mathrm{A}=\frac{2}{\sqrt[4]{5^5}}\cdot\frac{1+u^8}{u^2(1-u^8)^{\frac{1}{2}}}$$

which is a biquadratic in k.

(*iv*) To conclude, from A we obtain k and therefore q and ω. Thus y is obtained in terms of ϕ and finally x; the five values of x are

$$\frac{1}{\sqrt[4]{2^4.5^3}}\cdot\frac{\Phi(\omega)}{\phi(\omega)\psi^4(\omega)},\quad\frac{1}{\sqrt[4]{2^4.5^3}}\cdot\frac{\Phi(\omega+16)}{\phi(\omega)\psi^4(\omega)},$$

$$\frac{1}{\sqrt[4]{2^4.5^3}}\cdot\frac{\Phi(\omega+2.16)}{\phi(\omega)\psi^4(\omega)},\quad\frac{1}{\sqrt[4]{2^4.5^3}}\cdot\frac{\Phi(\omega+3.16)}{\phi(\omega)\psi^4(\omega)},$$

$$\frac{1}{\sqrt[4]{2^4.5^3}}\cdot\frac{\Phi(\omega+4.16)}{\phi(\omega)\psi^4(\omega)}.$$

In the above $\psi(\omega)=(k')^{\frac{1}{4}}$, $k'^{\frac{1}{2}}+k^2$ being equal to 1.

52. The comparison of different expressions of one and the same transcendental by means of theta-series has given many theorems about the representation of integers by quadratic forms. For example,

Jacobi proved Fermat's theorem, that every integer admits of being expressed as the sum of four squares. If the integer be $4n+1$ where n is an odd integer, then Jacobi proved that the number of different ways of expressing it as the sum of four squares is exactly equal to the sum of the factors of n.

53. The homogeneous co-ordinates x_1, x_2, x_3 of any curve of the nth degree and of deficiency unity admit of being expressed in terms of σ-products in the form

$$x_1 : x_2 : x_3 = \prod_{k=1}^{n} \sigma(u-a_{1,k}) : \prod_{k=1}^{n} \sigma(u-a_{2,k}) : \prod_{k=1}^{n} \sigma(u-a_{3,k})$$

of equal sums $\sum a_{1,k} = \sum a_{2,k} = \sum a_{3,k}$, and also in terms of the $\wp$-function as in

$$Rx_i = c_i + c_{i,0}\,\wp(u) + c_{i,1}\,\wp'(u) + \dots + c_{i,n-2}\,\wp^{(n-2)}(u),$$

where R is a proportionality-factor, the c's are constants and i takes the values 1, 2, 3.

In particular, if we take the triangle of reference to have a point of inflexion as one of its vertices and the opposite side as the harmonic polar of that vertex, then by suitably choosing the other two sides, we get the equation of the cubic in the forms

$$x_3{}^2 x_1 = x_2(x_1 - x_2)(x_1 - k^2 x_2)$$

$$x_3{}^2 x_1 = 4x_2{}^3 - g_2 x_2{}^2 x_1 - g_3 x_2 x_1{}^2 \, ;$$

and to the first equation corresponds the parametric representation $Rx_1 = \text{sn}^3 u$, $Rx_2 = \text{sn}\, u$, $Rx_3 = \text{sn}\, u \,\text{dn}\, u$ and to the second the representation

$$Rx_1 = 1,\ Rx_2 = \wp(u),\ R\,(x_3) = \wp'(u).$$

The use of elliptic functions helps greatly the study of the cubic curve, as has been shewn by many investigators including Halphen and Harnack.

If $x + iy = \text{sn}\,(u + iv)$, then the curves

$$u = \text{const},\quad v = \text{const},$$

form a system of bicircular quartics (see Sielvek's paper in *Crelle's Journal*, Vols. 54 and 59) ; similarly the relation

$$x+iy=\wp(u+iv)$$

gives a system of Cartesian ovals (see Greenhill's paper in *Proc. L.M.S.*, Vol. 17, 1st series).

54. An interesting geometrical application of elliptic functions is to the problem of the polygon of Poncelet, *viz.*, the problem of the determination, if possible, of a polygon which is at the same time inscribed in a conic and circumscribed to another conic. Poncelet showed that if, for two given conics and a given number as the number of the sides of the polygon, one polygon exists then there exists an infinite number. For the case when the conics are circles the problem was solved by Jacobi by means of the function am in 1828. (See *Crelle's Journal*, Vol. 3). The general case has been treated by Wirtinger and Loria.

55. I proceed to give Jacobi's solution on account of its historical importance :—

(*a*) Let the two given circles have centres C, c and radii R, r, respectively, the circle c being inside the circle C. Also let a denote the distance Cc. Then if, from any point A on the circle C, a tangent is drawn to the circle c it will cut the circle C again in A′; in the same manner the tangent drawn from A′ will cut the circle C again in A″, etc. Thus AA′A″A‴...will be a part of a polygon, or an unclosed polygon, which is inscribed in C and circumscribed round c.

Join c and C and let c C cut the circle C again in P. Then, denoting $\angle$ ACP by 2ϕ, $\angle$ A′CP by $2\phi'$, $\angle$ A″CP by $2\phi''$, etc., we have

$$\mathrm{R}\cos(\phi'-\phi)+a\cos(\phi'+\phi)=r,$$

$$\mathrm{R}\cos(\phi''-\phi')+a\cos(\phi''+\phi')=r,$$

$$\mathrm{R}\cos(\phi'''-\phi'')+a\cos(\phi'''+\phi'')=r,$$

$$\cdots \qquad \cdots \qquad \cdots$$

Hence

$$(\mathrm{R}+a)\cos\phi'\cos\phi+(\mathrm{R}-a)\sin\phi'\sin\phi=r,$$

$$(\mathrm{R}+a)\cos\phi''\cos\phi'+(\mathrm{R}-a)\sin\phi''\sin\phi'=r,$$

$$(\mathrm{R}+a)\cos\phi'''\cos\phi''+(\mathrm{R}-a)\sin\phi'''\sin\phi''=r,$$

$$\cdots \qquad \cdots \qquad \cdots$$

Subtracting each of these equations from the following, we have

$$\frac{\cos\phi''-\cos\phi}{\sin\phi''-\sin\phi}=-\frac{R-a}{R+a}\tan\phi',$$

$$\frac{\cos\phi'''-\cos\phi'}{\sin\phi'''-\sin\phi'}=-\frac{R-a}{R+a}\tan\phi'',$$

$$\ldots \qquad \ldots \qquad \ldots$$

Now, from the addition theorems for the functions, sn, cn, it is easily seen that the above equations are satisfied, if

$$\phi=\mathrm{am}(u),\quad \phi'=\mathrm{am}(u+t),\quad \phi''=\mathrm{am}(u+2t),\ldots,$$

where
$$\frac{R-a}{R+a}=\mathrm{dn}\, t,$$

the modulus k being given by

$$k^2=\frac{4Ra}{(R+a)^2-r^2}.$$

(*b*) If the polygon of n sides covers the circumference i times before becoming closed so that

$$\phi^{(n)}=\phi+2i\pi,$$

then
$$t=\frac{4iK}{n}, \qquad \ldots \quad (1)$$

where K, as usual denotes

$$\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}.$$

This equation is the necessary condition which must hold between R, r, a in order that a polygon of n sides may be inscribed in the circle C and circumscribed round the circle c.

56. The condition, in order that a polygon of n sides may be inscribed in a conic S and be circumscribed round another conic S′, is

easily expressible * as follows in terms of ξ_1, ξ_2, ξ_3, the roots of the equation obtained by equating the discriminant of $\xi S+S'$ to 0 :—

(*i*) If the roots be all positive and in descending order of magnitude, then the condition is

$$\xi_3=\xi_1 . \operatorname{cn}^2 \frac{2K}{n}, \quad \text{mod. } \sqrt{\frac{\xi_2-\xi_3}{\xi_1-\xi_3} . \frac{\xi_1}{\xi_2}}.$$

(*ii*) If one of the roots be negative, say ξ_3, and $\xi_1, < \xi_2$, then for an even n the condition is

$$\xi_2 \, k^2 \operatorname{cn}^2 \frac{2K}{n}=\xi_1 \operatorname{dn}^2 \frac{2K}{n}, \quad \text{mod. } \sqrt{\frac{\xi_2-\xi_3}{\xi_1-\xi_3} . \frac{\xi_1}{\xi_2}};$$

if n is odd, then the condition fails.

(*iii*) If the roots of the discriminating cubic are proportional to

$$1, \; re^{i\theta}, \; r e^{-i\theta},$$

then the required condition is

$$r=\frac{2 \operatorname{dn} \frac{4K}{n}}{1+\operatorname{cn} \frac{4K}{n}}, \quad \text{mod. } \cos \tfrac{1}{2} \tan^{-1} \frac{r \sin \theta}{1+r \cos \theta}.$$

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* See the paper of Rogers in *Proc. L. M. S.*, Series 1, Vol. 16.

SIXTH LECTURE

ABELIAN INTEGRALS AND ABELIAN FUNCTIONS.

57. I propose to introduce you in this lecture to certain entities more transcendental than the elliptic integrals and elliptic functions. These higher transcendentals, as I may call them, came into existence in 1825[*] when Abel succeeded in generalizing Euler's addition-theorem, *viz.*,—

If $\Phi(x)=\int_0^x \frac{dx}{\sqrt{\Phi(x)}}$, where Φ is a polynomial of the 4th degree in x, then y_1 can be always determined as an algebraic function of x_1 and x_2 so that

$$\Phi(x_1)+\Phi(x_2)=\Phi(y_1).$$

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[*] The generalization of Euler's theorem was given by Abel in his paper "Sur la comparison des fonctions transcendantes" which, although written in 1825, appeared after his death. The hyperelliptic case was published in *Crelle's Journal*, Vol. 3, in the paper "Remarques sur quelques propriétés générales d'une sorte de fonctions transcendantes" in 1828. The complete investigation of Abel on this subject is given in the memoir, "Sur une propriété générale d'une classe très étendue de fonctions transcendantes", which, although presented to the Academy of Sciences of Paris in 1826, was published by it in its memoirs as late as 1841. Although the importance of Abel's generalization was recognized by the mathematical world some years after Abel found it, Abel himself was fully conscious of the importance, as the following remarks from his memoir show. "The transcendental functions considered by mathematicians up to now have been very few in number. The theory of transcendental functions reduces almost to the theory of circular, logarithmic and exponential functions which are at the base all one. It is only in the last few years that other functions have began to be considered. Among them, the elliptic transcendentals, whose remarkable and elegant properties have been developed by M. Legendre, occupy the first rank. The author has considered, in the memoir which he has the honour to present to the Academy, a very extensive class of functions, *viz.* all those of which the derivative can be expressed by means of algebraic equations of which all the co-efficients are rational functions of one and the same variable, and he has found for those functions properties analogous to those of logarithmic and elliptic functions". An appreciation of Abel's generalization for the hyperelliptic case was published by Jacobi in his paper, "Considerationes generales de transcendentibus Abelienis", in *Crelle's Journal*, Vol. 9. Legendre called the theorem of Abel "monumentum aere perennius." Euler, Lagrange and other previous writers had failed to generalize Euler's theorem.

For example, if $\Phi(x)$ denote $\int_0^x \frac{(A+A_1x)}{\sqrt{\phi(x)}}dx$, where ϕ is a polynomial of the fifth or sixth degree and the A's are constants, then, according to Abel's generalization, it is always possible to find two quantities y_1 and y_2 as algebraic functions of x_1, x_2, x_3, so that

$$\Phi(x_1)+\Phi(x_2)+\Phi(x_3)=\Phi(y_1)+\Phi(y_2).$$

And, generally, if $\Phi(x)$ denote

$$\int_0^x \frac{(A_0+A_1x+A_2x^2+\ldots+A_{m-2}x^{m-2})dx}{\sqrt{\phi(x)}},$$

where $\phi(x)$ is a polynomial of the $2m$th or $(2m-1)$th degree, it is always possible to find $(m-1)$ quantities $y_1, y_2, \ldots y_{m-1}$ as algebraic functions of the m given quantities $x_1, x_2, \ldots x_m$, so that

$$\Phi(x_1)+\Phi(x_2)+\ldots+\Phi(x_m)=\Phi(y_1)+\Phi(x_2)+\ldots+\Phi(y_{m-1}).$$

The quantities y are the roots of an algebraic equation of the $(m-1)$th degree, in which each co-efficient is rationally expressible in terms of $x_1, x_2, \ldots x_m$, $\sqrt{\phi(x_1)}$, $\sqrt{\phi(x_2)}, \ldots \sqrt{\phi(x_m)}$.

58. The generalization of Euler's theorem is called *Abel's theorem* and may be enunciated as follows :—

Let y be the algebraic function of x given by an equation of the form

$$y^n+a_1y^{n-1}+a_2y^{n-2}+\ldots+a_n=0,$$

where $a_1, a_2, \ldots, a_n$ are polynomials in x, and the left side of the equation is incapable of resolution into factors of the same rational form ; also let $R(x, y)$ be any rational function of x and y. Then the sum of any number of similar integrals

$$\int^{(x_1,y_1)}R(x,y)dx+\int^{(x_2,y_2)}R(x,y)dx+\ldots+\int^{(x_m,y_m)}R(x,y)dx,$$

with arbitrary lower limits, is expressible by rational functions of $(x_1, y_1), \ldots (x_m, y_m)$, and logarithms of such rational functions, with the addition of the sum of a certain number, n, of integrals

$$-\int^{(z_1,s_1)}R(x, y)dx-\ldots-\int^{(z_n,s_n)}R(x, y)dx,$$

wherein $z_1, \dots z_n$ are values of x, determinable from $x_1, y_1, \dots x_m, y_m$ as the roots of an algebraic equation whose co-efficients are rational functions of $x_1, y_1, \cdots x_m, y_m$, and $s_1, \dots s_n$ are the corresponding values of y, of which any one, say s_i, is determinable as a rational function of z_i and $x_1 y_1, \dots x_m, y_m$.

59. The integrals, considered in the preceding article are called *Abelian integrals* ; the special cases of these which we considered in Art. 57 are called *hyperelliptic integrals*.

Abelian integrals are of three kinds. If $R(x, y)$ has only the branch-points as poles then the integral can always remain finite and we have the integral of the *first kind*. If $R(x, y)$ has the point (x_0, y_0) as a pole of higher order than the first, whose residue is zero, then the integral becomes infinite in (x_0, y_0) as an algebraic function and is called an Abelian integral of the *second kind*. Finally, if R has a point (x_0, y_0) as a pole of the first order, then the integral becomes infinite in this point as the logarithm of an algebraic function and is called an integral of *the third kind*.

Hyperelliptic integrals ara also of three kinds. The hyperelliptic integral

$$\int^x \frac{dx}{\sqrt{\phi(x)}}$$

is said to be of *the first kind* if it does not become infinite at any point ; of *the second kind* if it becomes infinite only as an algebraic function ; and of *the third kind* if it becomes infinite only as a logarithm.

A hyperelliptic integral is said to be of *the* $(m-1)$*th order* if the degree of $\phi(x)$ is the $(2m-1)$th or $2m$th.

60. (1) With every Abelian integral are associated at least an even number, (say $2p$) of constants, called the *periodicity-moduli* of the integral ; they possess the property that the difference of any two values which the integral can have is equal to the sum of integral multiples of its moduli.

(2) The integrals of the first two kinds have only the aforesaid $2p$ moduli ; those of the third kind have in addition others which depend on the points of logarithmic infinities, *i.e.*, if one goes round one of these points then the value of the integral increases by $2\pi i$ multiplied by an integral multiple of the residue corresponding to the point.

(3) The sum of the residues of every integral of the third kind is zero.

(4) Every Abelian integral admits of being formed by a linear combination of integrals of the three kinds.

(5) There exist p Abelian integrals of the first kind which are linearly independent of one another, if the Riemann's surface used is irresolvable and of the genus p.

(6) If the algebraic equation, whose root is y, is resolvable into k factors, then there are $p-k+1$ linearly independent Abelian integrals.

(7) The periodicity-moduli of an integral of the first kind cannot be all real or all imaginary.

61. The problem of the reduction of hyperelliptic integrals to elliptic integrals first engaged the attention of Legendre, who, in the third part of his " Traité des fonctions elliptiques," shows how

$$\int_0^x \frac{dx}{\sqrt{x(1-x^2)(1-n^2x^2)}}$$

can be expressed as the sum of two elliptic integrals of the first kind whose amplitude is the same and whose moduli are complementary to each other.

In *Crelle's Journal*, Vol. 8, 1832, Jacobi generalized the above result of Legendre and showed that

$$\int_0^x \frac{dx}{\sqrt{x(1-x)(1-k\lambda x)(1+kx)(1+\lambda x)}}$$

can also be expressed as the sum of two elliptic integrals of the first kind whose amplitude is the same but whose moduli are not always complementary to each other ; if k and λ are suitably chosen, the moduli may be arbitrary.

In his paper, " Sur un exemple de reduction d' integrales abelienes aux fonctions elliptiques " (*Ann. Soc. Scient. Bruxelles*, 1876 ; *Oeuvres*, t. 3, pp. 249-261), Hermite gave another example of a reduction similar to Jacobi's. It is this : If

$$ax=4z^3-3az,\ 3y(z^2-a)=2z^3-b,$$

$$S^2=(z^2-a)(8z^3-6az+b),$$

then we have

$$\int\frac{dz}{S}=\frac{1}{3}\int\frac{dx}{\sqrt{(2ax-b)(x^3-a)}},\quad \int\frac{z\,dz}{S}=\frac{1}{2\sqrt{3}}\int\frac{dy}{\sqrt{y^3-3ay-b}}.$$

Koenigsberger has, in his paper, "Reduktion ultraelliptischer Integrale auf elliptische" (*Crelle's Journal*, Vol. 67, 1867) considered still more general cases and finds that there is a possibility of reduction in the case of a hyperelliptic integral, if the polynomial of the sixth degree under the radical sign, when equated to 0, furnishes three pair-points of an involution.

In addition to the integrals

$$\int\frac{dz}{\sqrt{z^6+az^4+bz^2+c}},\ \int\frac{z\,dz}{\sqrt{z^6+az^4+bz^2+c}},$$

which are obviously reducible to elliptic integrals by the substitution $z^2=x$, the two integrals

$$\int\frac{dz}{\sqrt{(z^3+az+b)(z^3+pz^2+q)}},\ \int\frac{z\,dz}{\sqrt{(z^3+az+b)(z^3+pz^2+q)}},$$

where

$$q=4b+\frac{4}{3}ap,$$

reduce to elliptic integrals by the substitutions

$$x=\frac{z^3+az+b}{3z-p},\quad x=\frac{z^3+pz^2+q}{az^3-3bz^2},$$

respectively (see *Bulletin de la Société Mathematique de France*, t. XII).

Examples:

$$\int\frac{dz}{\sqrt{a+\gamma z^3+\beta z^6}}\ \text{ and }\ \int\frac{z\,dz}{\sqrt{a+\gamma z^3+\beta z^6}}$$

are reducible to elliptic integrals when a and β have the same sign.

62. It was again Legendre who first gave examples of the reduction of Abelian integrals other than hyperelliptic integrals to elliptic integrals. Roethig has considered these carefully in his paper, "Ueber

einige Gattungen elliptischer Integrale" (*Crelle's Journal*, Vol. 56). He shows (α) that

$$\int \frac{f(x)dx}{(a+a_1x+a_2x^2+a_3x^3)^{\frac{m}{3}}},$$

wherein $f(x)$ is a rational function and m is any integer of the form $3n+1$ or $3n+2$, can always be reduced to elliptic integrals with the modulus $\frac{1}{2}\sqrt{2\pm\sqrt{3}}$, and ($\beta$) that the integral of the form

$$\int \frac{f(x)dx}{(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4)^{\frac{m}{4}}},$$

wherein m is of the form $4n+1$ or $4n+3$, admits of being reduced to elliptic integrals with the modulus $\sqrt{\tfrac{1}{2}}$.

Briot and Bouquet have considered in their book, "Théorie des fonctions elliptiques," the question of the reduction of an Abelian integral, whose irrationality is given by a binomial equation, on the supposition that the upper limit of the integral is everywhere a single-valued function, and shown that only nine integrals have the property of reduction; they are

$$\int \frac{dx}{\sqrt[3]{(x-a)^2(x-b)^2}}, \quad \int \frac{dx}{\sqrt[3]{(x-a)^2(x-b)^2(x-c)^2}},$$

$$\int \frac{dx}{\sqrt[4]{(x-a)^2(x-b)^3}}, \quad \int \frac{dx}{\sqrt[4]{(x-a)^2(x-b)^3(x-c)^3}},$$

$$\int \frac{dx}{\sqrt[4]{(x-a)^3(x-b)^3}} \quad \int \frac{dx}{\sqrt[6]{(x-a)^3(x-b)^4}},$$

$$\int \frac{dx}{\sqrt[6]{(x-a)^5(x-b)^3}}, \quad \int \frac{dx}{\sqrt[6]{(x-a)^4(x-b)^5}},$$

$$\int \frac{dx}{\sqrt[6]{(x-a)^3(x-b)^4(x-c)^5}},$$

This question has been investigated also by Koenigsberger, Krazer, Weierstrass and many others.

The following question has not ~~yet~~ been fully answered :

What are the cases in which for a given Abelian integral

$$\int R(x, y)dx$$

there exists an algebraic substitution which changes this integral into a new Abelian integral with a smaller number of periodicity-moduli ?

It can be shown that when the $2p$ periodicity-moduli of the given integral reduce to two distinct moduli, the integral can be reduced to an elliptic integral by a rational substitution.

63. Denoting by p, q, R polynomials in x, Abel considered the following problem : "To find all the differentials of the form $\frac{\mathfrak{C}dx}{\sqrt{R}}$, where $\mathfrak{C}$ and R are polynomials in x, of which the integrals can be expressed in the form

$$\log \frac{p+q\sqrt{R}}{p-q\sqrt{R}} ."$$

The complete solution of this problem was given by Abel as follows :

"When it is possible to find R corresponding to $\mathfrak{C}$, a given polynomial, such that

$$\int \frac{\mathfrak{C}dx}{\sqrt{R}} = \log \frac{y+\sqrt{R}}{y-\sqrt{R}},$$

then the continued fraction for $\sqrt{R}$ is periodic and of the form

$$\sqrt{R} = r + \frac{1}{2\mu +}\ \frac{1}{2\mu_1 + \ldots}\ \frac{1}{2\mu_1 +}\ \frac{1}{2\mu +}\ \frac{1}{2r +}\ \frac{1}{2\mu +}\ \frac{1}{2\mu_1 + \ldots},$$

and, conversely, when the continued fraction of a given $\sqrt{R}$ has this form then it is always possible to find a polynomial $\mathfrak{C}$ for which

$$\int \frac{\mathfrak{C}dx}{\sqrt{R}} = \log \frac{y+\sqrt{R}}{y-\sqrt{R}} .$$

The function y is given by

$$y=r+\frac{1}{2\mu+}\;\frac{1}{2\mu_1+}\;\frac{1}{2\mu_2+\ldots}\;\frac{1}{2\mu+}\;\frac{1}{2r}."$$

Abel also proved the following theorem : Whenever an integral of the form $\int\frac{\mathfrak{C}\,dx}{\sqrt{R}}$, where $\mathfrak{C}$ and R are polynomials in x, is expressible by logarithms, one can always express the integral in the form

$$\int\frac{\mathfrak{C}\,dx}{\sqrt{R}}=A\log\frac{p+q\sqrt{R}}{p-q\sqrt{R}},$$

where A is a constant, and p and q are polynomials in x.

The question of the reduction of certain kinds of Abelian integrals to logarithms does not offer much difficulty. The following examples* may be regarded as illustrations ;

$$(1)\quad \int\frac{x\,dx}{\sqrt[3]{(x^3-1)^2}}=-\frac{1}{3}\log\Big\{(x-\sqrt[3]{x^3-1})\,(x-a\sqrt[3]{x^3-1})^a$$

$$\times(x-a^2\sqrt[3]{x^3-1})^{a^2}\Big\},$$

a being an imaginary cube root of unity.

$$(2)\quad \int\frac{dx}{\sqrt[3]{x^3-px}}=-\frac{1}{2}\log\Big\{x^{\frac{2}{3}}-\sqrt[3]{x^2-p})\,(x^{\frac{2}{3}}-a\sqrt[3]{x^2-p})^{a^2}$$

$$\times(x^{\frac{2}{3}}\quad a^2\sqrt[3]{x^2-p})^a\Big\}.$$

$$(3)\quad \int\frac{dx}{\sqrt[3]{x^3-p}}=-\frac{1}{3}\log\Big\{(x-\sqrt[3]{x^3-p})\,(x-a\sqrt[3]{x^3-p})^{a^2}$$

$$\times(x-a^2\sqrt[3]{x^3-p})^a\Big\}.$$

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* J. Dolbnia, "Sur les integrales pseudo-elliptiques qui dependent d'une racine cubique d'un polynome du troisieme degré" (*Bulletin des Sciences Mathematiques*. 1893).

(4) $\int \frac{dx}{\sqrt[3]{(x-a)(x-b)(x-c)}}$ is expressible as a logarithm ; if

$z_0 = \frac{2\omega}{m}$ is a commensurable part of a period of $\wp$, where

$g_2 = 0$, $g_3 = -(b-c)^2/3^6(a-b)^3(a-c)^3$ and $\wp(z_0) = 1/9\sqrt[3]{(a-b)(a-c)}$.

64. Soon after the publication of Abel's theorem, Jacobi placed before himself the problem of introducing into Analysis the inverse-function of a hyperelliptic integral in the same manner as the elliptic functions had been introduced as inverse functions of elliptic integrals. But in his attempt to solve this problem, Jacobi had to face this serious difficulty : as the hyperelliptic integral possesses four or more periods, its inverse function must necessarily possess the same number of periods, but such an inverse function cannot be single-valued as a single-valued function cannot have more than two periods. This difficulty was overcome by Jacobi by his realizing that the inverse function, if considered as a function of p variables, may be single-valued and still have $2p$ independent periods.

The more general case of Abelian integrals, other than hyperelliptic integrals, was not attempted for inversion before **1844** when Hermite wrote his memoir, " Sur la the'orie des transcendantes a diffe'rentielles algebriques" (*Comptes Rendus*, t, 18).

65. (*a*) I proceed first to give in the words of Weierstrass* the definition of a particular case of an Abelian function, *viz.*, a hyperelliptic function :—

" The theorem of Abel on hyperelliptic integrals forms the foundation for the theory of a new class of analytical functions, which may for that reason be suitably called Abelian functions, and which may be defined as follows.

Let $R(x)$ denote $A_0(x-a_1)(x-a_2)\ldots\left(x-a_{2\rho+1}\right)$, an integral function of x of the $(2\rho+1)$th degree, in which it is assumed that of the quantities $a_1, a_2, \ldots a_{2\rho+1}$ no two are equal although they otherwise may take any real or imaginary values. Further, let $u_1, u_2, \dots u_\rho$ be ρ unrestrictedly variable quantities and between these and an equal number of quantities $x_1, x_2, \dots x_\rho$ dependent on them let the following differential equations hold :

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* "Theorie der Abelschen Functionen,' (*Crelle's Journal*, Bd. 52, 1856, pp. 285 and 286).

$$du_1 = \frac{1}{2}\frac{P(x_1)}{x_1 - a_1}\frac{dx_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2 - a_1}\frac{dx_2}{\sqrt{R(x_2)}} + \dots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho - a_1}\frac{dx_\rho}{\sqrt{R(x_\rho)}},$$

$$du_2 = \frac{1}{2}\frac{P(x_1)}{x_1 - a_2}\frac{dx_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2 - a_2}\frac{dx_2}{\sqrt{R(x_2)}} + \dots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho - a_2}\frac{dx_\rho}{\sqrt{R(x_\rho)}},$$

$$\dots\dots\dots \qquad \dots\dots \qquad \dots\dots$$

$$du_\rho = \frac{1}{2}\frac{P(x_1)}{x_1 - a_\rho}\frac{dx_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2 - a_\rho}\frac{dx_2}{\sqrt{R(x_2)}} + \dots$$

$$+ \frac{1}{2}\frac{P(x_\rho)}{x_\rho - a_\rho}\frac{dx_\rho}{\sqrt{R(x_\rho)}},$$

where $P(x)$ stands for $(x - a_1)(x - a_2)\dots(x - a_\rho)$ and it is understood that $x_1, x_2 \dots x_\rho$ take the values $u_1, u_2, \dots u_\rho$ if the quantities $u_1, u_2, \dots u_\rho$ all vanish. Then $x_1, x_2 \dots x_\rho$ are to be considered as the roots of an equation of the form

$$x^\rho + P_1 x^{\rho-1} + P_2 x^{\rho-2} + \dots P_\rho = 0,$$

where $P_1, P_2, \dots P_\rho$ denote single-valued analytical functions of $u_1, u_2 \dots u_\rho$; whilst a second integral function of x of the $(\rho-1)$th degree

$$Q_1 x^{\rho-1} + Q_2 x^{\rho-2} + \dots + Q_\rho,$$

whose co-efficients are also such functions of $u_1, u_2, \dots u_\rho$, gives respectively the values

$$\sqrt{R(x_1)}, \sqrt{R(x_2)}, \dots \sqrt{R(x_\rho)}$$

when x is put equal to $x_1, x_2, \dots x_\rho$.

After this, every expression, formed rationally and symmetrically out of $x_1, x_2, \dots x_\rho$ and $\sqrt{R(x_1)}, \sqrt{R(x_2)}, \dots \sqrt{R(x_\rho)}$, is to be looked upon as a single-valued function of $u_1, u_2, \ldots u_\rho$. In particular, however, it follows that the product

$$(a_r - x_1)(a_r - x_2)\ldots(a_r - x_\rho),$$

where r denotes any one of the integers $1, 2, \ldots 2\rho+1$, is the square of such a one. Accordingly, if one considers the quantities

$$\sqrt{h_1\, \phi\,(a_1)},\quad \sqrt{h_2\, \phi\,(a_2)}, \ldots \sqrt{h_{2\rho+1}\, \phi(a_{2\rho+1})}$$

as functions of $u_1, u_2, \ldots u_\rho$, $\phi(x)$ standing for $(x-x_1)(x-x_2)\ldots\ldots(x-x_\rho)$ and $h_1, h_2, \ldots h_{2\rho+1}$ being constants; then, out of those quantities, cannot only the co-efficients of the equation whose roots are $x_1, x_2, \ldots x_\rho$ be formed easily but they (the quantities) are also marked, like the elliptic functions sin *am* u, cos *am* u, Δ *am* u, to which they are reduced for $\rho=1$, and to which they are altogether analogous, by such a huge number of remarkable and fruitful properties that one is justified in giving them and a number of others, connected with them, by preference the name '*Abelian functions*' and is encouraged to make them the chief subject of consideration."

The general Abelian function may be defined as follows:

Let $u_1, u_2, \cdots u_p$ denote linearly independent Abelian integrals of the first kind and let

$$U_h^x \equiv \sum_{i=1}^{p} \int_{a_i^0}^{x_i} du_h, \qquad (h=1, 2, \ldots p);$$

and further let $S(x_1, x_2, \ldots x_p)$ denote any rational and symmetric function of $x_1, x_2, \ldots x_p$. Then the function S considered as expressed by means of the quantities U_h is called an Abelian function of the arguments U_h and is denoted by the symbol

$$Al(U_1, U_2, \ldots U_p).$$

66. The general Abelian function may also be defined as follows:—

A single-valued function $f(v_1, v_2, \ldots v_p)$ of p variables $v_1, v_2, \ldots v_p$, which has everywhere in the finite domain the character of a rational function, is called an Abelian function, if it is $2p$-ply periodic, *i.e.*, if there exist p linearly and integrally independent system of quantities

$$(1) \qquad \omega_{1k},\ \omega_{2k},\ldots\omega_{pk}, \qquad (k=1,\ 2,\ldots 2p)$$

so that

$$f(v_1+\omega_{1k},\ v_2+\omega_{2k},\ldots v_p+\omega_{pk})=f(v_1,\ v_2,\ldots v_p).$$

The p quantities $\omega_{1k},\ \omega_{2k},\ldots\omega_{pk}$ are said to constitute a *system of periods* of the function $f(v_1,\ v_2,\ldots v_p)$.

If $n_1,\ n_2,\ldots n_{2p}$ are integers and

$$(2) \quad \omega_k=n_1\omega_{k1}+n_2\omega_{k2}+\ldots+n_{2p}\omega_{k2p}, \quad (k=1,\ 2,\ldots p)$$

then is also

$$f(v_1+\omega_1,\ v_2+\omega_2,\ldots v_p+\omega_p)=f(v_1,\ v_2,\ldots v_p).$$

The quantity ω_k is, therefore, also a period of $f(v_1,\ v_2,\ldots v_p)$ or briefly $f(v)$.

An Abelian function has, therefore, infinitely many periods. But quantities $\omega_{k,l}$ admit of being so chosen that every period is representable in the form (2) with integral n's. Such a system of periods $\omega_{k,l}$ is called *primitive*.

All Abelian functions which have the quantities $\omega_{k,l}$ as periods are said to form a *class*.

67. The following are the principal properties of the Abelian functions:

(1) The differential co-efficient of an Abelian function is an Abelian function of the same class.

(2) Corresponding to each Abelian function $f_1(v)$ one can determine $p-1$ functions $f_2(v),\ f_3(v),\ldots f_p(v)$ of the same class, such that their functional determinant is not zero identically. The equations

$$f_1(v)=s_1,\ f_2(v)=s_2,\ldots f_p(v)=s_p,$$

have then, if the s's are not specially chosen, only a finite number of incongruent solutions; this number, which may be denoted by m, is *independent* of the choice of the s's provided that they are not singular.

(3) If $f(v)$ be any new function of the same class as the functions $f_1(v), f_2(v)\ldots f_p(v)$, then it satisfies an algebraic equation of degree m, whose co-efficients are rational functions of $f_1(v), f_2(v),\ldots f_p(v)$.

(4) If the aforesaid equation of the mth degree is irreducible, then all the functions of the class admit of being expressed rationally by the $p+1$ functions $f_1(v), f_2(v), \ldots f_p(v)$.

(5) Every Abelian function admits of being represented as the quotient of two integral transcendental functions.

(6) An Abelian function is never an integral transcendental function.

(7) Every Abelian function possesses an algebraic addition-theorem.

(8) Every Abelian function admits of being expressed rationally by means of $p+1$ suitable Abelian functions, particularly by means of an Abelian function and its p differential co-efficients of the first order.

(9) Between an Abelian function and its p differential co-efficients of the first order, there exists an algebraic relation.

68. I proceed now to give you an idea of the explicit representation of an Abelian function in terms of its arguments, by taking the very special case of a hyperelliptic function.

(*a*) Let s denote

$$\sqrt{(x-c_1)(x-c_2)(x-c_3)(x-c_4)(x-c_5)(x-c_6)}$$

and let

$$w_1(x,s) \equiv w_1 = \int \frac{dx}{s}; \quad w_2(x,s) \equiv \omega_2 = \int \frac{x\,dx}{s}.$$

Then every hyperelliptic integral of the first kind can be expressed in terms of w_1 and w_2 by combining them linearly with constant coefficients. Let the periodicity-moduli of w_1 be denoted by A_{1k}, A_{2k} and those of w_2 by B_{1k}, B_{2k}. Then

$$\int_{c_1}^{c_2} dw_1 = \frac{1}{2} A_{11}, \int_{c_2}^{c_1} dw_1 = -\frac{1}{2} B_{11}, \quad \int_{c}^{c_4} dw_1 = -\frac{1}{2} A_{12},$$

$$\int_{c_5}^{c_6} dw_1 = -\frac{1}{2} B_{12},$$

$$\int_{c_1}^{c_2} dw_2 = \frac{1}{2} A_{21}, \quad \int_{c_0}^{c_1} dw_2 = -\frac{1}{2} B_{21}, \quad \int_{c_3}^{c_4} dw_2 = -\frac{1}{2} A_{22}$$

$$\int_{c_5}^{c_0} dw_2 = -\frac{1}{2} B_{22},$$

Further, let

$a_{l,k} = \dfrac{d(\log | A |)}{d\, A_{l,k}}$, where $| A |$ stands for the determinant $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$,

$$u_1 = a_{11}\, w_1 + a_{21}\, w_2, \quad u_2 = a_{21}\, w_1 + a_{22} w_2 ;$$

$$\tau_{lk} = a_{1l}\, B_{1k} + a_{2l}\, B_{2k} = \tau_{kl} = a_{1k} B_{1l} + a_{2k} B_{2l}.$$

(*b*) Let $\theta^{h_1 h_2}_{g_1 g_2}(v)$ denote the theta-function of two variables v_1, v_2, viz.,

$$\theta^{h_1 h_2}_{g_1 g_2}(v) \equiv$$

$$\sum_{m_1} \sum_{m_2} e^{\frac{1}{4} i\pi [\tau_{11}(2m_1 - h_1)^2 + 2\tau_{12}(2m_1 - h_1)(2m_2 - h_2) + \tau_{22}(2m_2 - h_2)^2]} \times \cos \Omega_{m_1, m_2},$$

where g_1, g_2, h_1, h_2 are integers,

$$\Omega_{m_1 m_2} = \pi(v_1 - \tfrac{1}{2} g_1)(2m_1 - h_1) + \pi(v_2 - \tfrac{1}{2} g_2)(2m_2 - h_2),$$

and the summation extends to all integral values of m_1 and m_2 from $-\infty$ to ∞.

Then the hyperelliptic function x can be expressed as an algebraic function of quotients of theta-functions, *e.g.*,

$$(x - c_1)^4 (x - c_3)^4 / (x - c_2)^4 (x - c_5)^4$$

$$= \{\theta^{01}_{10}(u)\}^8 \Pi_p^{1,3} (c_1 - c_p)^2 (c_3 - c_p)^2 / \{\theta^{11}_{10}(u)\}^8 \Pi_p^{1,3} (c_1 - c_p)^2 (c_2 - c_p)^2,$$

where $\overset{\mu\mu'}{\Pi}_p (c_\mu - c_p)^2 (c_{\mu'} - c_p)^2$ is a product of factors of the form

$$(c_\mu - c_p)(c_{\mu'} - c_p)$$

if p takes all the values from 1 to 6, excepting μ and μ'.

69. The applications of the transcendentals, discussed by me in to-day's lecture, are varied and many ; but, for want of time, I shall content myself with giving only two applications, one to Dynamics and the other to Differential Geometry.

(*a*) Consider a particle constrained to move on the ellipse

$$(x-1)^2+\frac{y^2}{1-e^2}=1$$

with the axis of x vertical. Then the principle of vis viva gives

$$v^2=2g\,(h-x),$$

and, consequently,

$$dt=\frac{\{1-e^2(1-x)^2\}dx}{e\sqrt{2g}\;s},$$

where s denotes

$$\sqrt{x\left(\frac{1-e}{e}+x\right)(h-x)(2-x)\left(\frac{1+e}{e}-x\right)}$$

Therefore the time occupied by the particle in moving from $x=0$ is

$$\frac{1}{e\sqrt{2g}}\int_0^x\frac{\{1-e^2(1-x)^2\}dx}{s},$$

a hyperelliptic integral of the second kind ; since, for large values of x,

$$\frac{1-e^2(1-x)^2}{s}=\frac{A}{\sqrt{x}}+\frac{B}{(\sqrt{x})^3}+\frac{C}{(\sqrt{x})^5}+\cdots$$

and, therefore, the integral behaves at $x=\infty$ as $\sqrt{x}$, an algebraic function.

(*b*) Weierstrass[*] has shown that the rectangular co-ordinates of the point on a geodesic line on an ellipsoid of three unequal axes can be expressed by means of hyperelliptic functions and that the length of the line is expressible as a hyperelliptic integral of the second kind.

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[*] Ueber die geodaetische Linien auf dem dreiaxigen Ellipsoid" *Monatsbericht der Koeniglichen Akademie der Wissenschaften zu Berlin, 1861*).

APPENDIX A.

Expansions of $\sigma(z)$ and $\wp(z)$ in powers of z.

1. As stated in the fourth lecture,* Weierstrass obtained the expansion of $\sigma(z)$ up to z^{35}; the co-efficients $a_{m,n}$ are given below in tabular form :— †

$z^{4m+6n+1}$	m	n	$a_{m,n}$
z^1	0	0	+1
z^5	1	0	−1
z^7	0	1	−3
z^9	2	0	−9
z^{11}	1	1	−18
z^{13}	0	2	−54
...	3	0	+69
z^{15}	2	1	+513
z^{17}	4	0	+321
...	1	2	+4968
z^{19}	0	3	+14904
...	3	1	+33588
z^{21}	2	2	+257580
...	5	0	+160839
z^{23}	1	3	+502200
...	4	1	+2808945
z^{25}	0	4	+1506600
...	3	2	+20019960
...	6	0	+1416951
z^{27}	2	3	+162100440
...	5	1	−41843142
z^{29}	1	4	+796330440
...	4	2	−376375410
...	7	0	−388946691
z^{31}	0	5	+2388991320
...	3	3	−9465715080
...	6	1	−6519779667
z^{33}	2	4	−144916218720
...	5	2	−210469286736
...	8	0	+25514578881
z^{35}	1	5	−1289959784640
...	4	3	−4582619446320
...	7	1	−485174610648

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* See Art. 43, pp. 42-43.

† See H. A. Schwarz : "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen, nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass" (Göttingen, 1883 ; 2nd edition, Berlin, 1893), p. 7.

2. In the third lecture, the recurrence formula for the coefficients c_κ in the expansion

$$\wp(z)=\frac{1}{z^2}+\sum_{2}^{\infty} c_\kappa \, z^{2\kappa-2} \quad ,$$

has been given.

By using this formula, Mr. Subodh Chandra Mitra has obtained the expansion of $\wp(z)$ up to z^{36}, thus obtaining nine more terms of the expansion than had been obtained by any previous writer.* The values of the various coefficients from c_2 up to c_{19} are given below :—

$$c_2=\frac{g_2}{2^2.5}, \quad c_3=\frac{g_3}{2^2.7}, \quad c_4=\frac{g_2^2}{2^4.3.5^2}, \quad c_5=\frac{3g_2g_3}{2^4.\,5.7.11}$$

$$c_6=\frac{1}{13}\left\{\frac{g_2^3}{2^4.3.5^3}+\frac{g_3^2}{2^4.7^2}\right\}, \quad c_7=\frac{g_2^2g_3}{2^3.3.5^2.7.11},$$

$$c_8=\frac{1}{17}\left\{\frac{g_2^4}{2^5.3.5^3.13}+\frac{3g_2g_3^2}{2^4.7^2.11.13}\right\},$$

$$c_9=\frac{1}{19}\left\{\frac{29g_2^3g_3}{2^5.5^3.7.11.13}+\frac{g_3^3}{2^6.7^3.13}\right\},$$

$$c_{10}=\frac{1}{21}\left\{\frac{7g_2^5}{2^9.3.5^5.13.17}+\frac{97g_2^2g_3^2}{2^8.5.7.11^2.13.17}\right\},$$

$$c_{11}==\frac{1}{23}\left\{\frac{389g_2^4g_3}{2^9.3.5^4.11.13.17.19}+\frac{123g_2g_3^3}{2^7.5.7^3.11.13.17.19}\right\},$$

$$c_{12}=\frac{1}{25}\left\{\frac{g_2^6}{2^{11}.3^3.5^5.13^2.17}+\frac{25g_2^3g_3^2}{2^6.7^2.11^2.13^2.17.19}\right.$$

$$\left.+\frac{15g_3^4}{2^8.7^4.13^2.19}\right\},$$

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* See Mitra's paper : "On the expansions of the Weierstrassian and Jacobian elliptic functions in powers of the argument" (*Bulletin of the Calcutta Mathematical Society*, Vol. 17, 1926, pp. 159-172).

$$c_{13}=\frac{1}{27}\left\{\frac{729g_2^5g_3}{2^{11}.5^4.7.11.13.17.19.23}+\frac{1431g_2^2g_3^3}{2^8.7^3.11^2.13.17.19.23}\right\},$$

$$c_{14}=\frac{1}{29}\left\{\frac{g_2^7}{2^{13}.3.5^7.13^2.17}+\frac{13647g_2^4g_3^2}{2^{11}.5^3.7.11^2.13^2.17.19.23}\right.$$

$$\left.+\frac{6471g_2g_3^4}{2^8.5^2.7^3.11.13^2.17.19.23}\right\},$$

$$c_{15}=\frac{1}{31}\left\{\frac{104003g_2^6g_3}{2^{13}.3^3.5^6.7.11.13^2.17.19.23}\right.$$

$$\left.+\frac{399701g_2^3g_3^3}{2^9.3.5^4.7^3.11^3.13^2.17.19.23}+\frac{g_3^5}{2^8.5.7^5.13^2.19}\right\},$$

$$c_{16}=\frac{1}{33}\left\{\frac{2453g_2^8}{2^{16}.3^3.5^8.13^2.17^2.29}+\frac{1006029g_2^5g_3^2}{2^{11}.5^6.7^2.11.13^2.17^2.19.23.29}\right.$$

$$\left.+\frac{8105017g_2^2g_3^4}{2^{10}.5^3.7^4.11.13^2.17^2.19.23.29}\right\},$$

$$c_{17}=\frac{1}{35}\left\{\frac{49871g_2^7g_3}{2^{16}.3^2.5^6.13^2.17.19.23.29.31}\right.$$

$$+\frac{240263g_2^4g_3^3}{2^{14}.3.5^2.7.11^2.13^3.17.19.23.29.31.}$$

$$\left.+\frac{3693g_2g_3^5}{2^{12}.5.7^3.13^2.17.19.23.29.31}\right\},$$

$$c_{18}=\frac{1}{37}\left\{\frac{427g_2^9}{2^{17}.3^3.5^8.13^3.17^2.29}+\frac{30458088737g_2^6g_3^2}{2^{16}.3^2.5^5.7.11^2.13^3.17^2.19^2.23.29.31.}\right.$$

$$\left.+\frac{122378650673g_2^3g_3^4}{2^{15}.5^3.7^3.11^3\,13^3.17^2.19^2\,23.29.31}+\frac{43g_3^6}{2^{12}.7^5.13^3.19^2.31}\right\},$$

$$c_{19}=\frac{1}{39}\left\{\frac{5562883g_2^8g_3}{2^{17}.3^3.5^4.7.11.13.17^2.19.23.29.31}\right.$$

$$+\frac{429852433g_2^5g_3^3}{2^{15}.3.5^5.7^3.11^2.13.17^2.19.23.29.31}$$

$$\left.+\frac{28712941g_2^2g_3^5}{2^{18}.5^3.7^5.11^2.17^2.19.23.29.31}\right\}.$$

3. In a communication[*] to the Academy of Sciences of Berlin in 1882, Weierstrass states, as already given in Schwarz's "Formeln und Lehrsaetze," the values of the coefficients $a_{m,n}$ up to those for z^{35}. In a paper,[†] communicated to the Society of Sciences of Gœttingen, Haussner attempted to express these coefficients independently as determinants.

4. With reference to the function $\wp(z)$, attempts were made by a number of writers, including Hurwitz and Herglotz[‡] to find a general expression for c_N by using which the values of the c's for all values of N can be written down.

Hurwitz confines his investigation to the special case in which g_3 is zero and $g_2=4$, and finds that

$$\wp(z)=\frac{1}{z^2}+\sum_{n=1}\frac{2^{4n}E_n}{4n}\cdot\frac{z^{4n-2}}{(4n-2)!},$$

where the E's are certain numbers analogous to Bernoulli's numbers. Hurwitz fails to give a simple expression for E_n from which all the E's can be easily calculated. But he shows that E_n is of the form

$$G_n+\frac{1}{2}+\sum{}'\frac{(2a)^{\frac{4n}{p-1}}}{p},$$

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[*] "Zur Theorie der elliptischen Functionen" (*Sitzungsberichte d. k. Preuss. Akademie d. Wissenschaften* zu Berlin, 1882, p. 251)

[†] "Ueber die Zahlencoefficienten in den Weierstrass'schen, σ-Reihen." (*Nachrichten der k. Gesellschaft d. Wissenschaften* zu Gœttingen, 1894).

[‡] Hurwitz, *Gœtt. Nachrichten*, 1897, pp. 273-275; Herglotz, *Leipziger Berichte* 1922, pp. 269-289.

where G_n is an *undetermined* odd integer, p denotes a prime number of the form $4k+1$, a^2 is the odd square of the two squares in which p can be broken so that

$$p=a^2+b^2,$$

a is to be taken with such a sign that the congruence

$$a\equiv b+1 \pmod{4}$$

holds, and lastly the summation for p extends to only those primes for which $p-1$ is a divisor of $4n$.

APPENDIX B

THETA FUNCTIONS OF ONE OR MORE VARIABLES.

1. In view of the fact that Jacobi built up a theory of elliptic functions, based on the four theta functions, defined in Art. 25 of my second lecture, it is desirable to introduce three functions, $\sigma_1(z)$, $\sigma_2(z)$, $\sigma_3(z)$, anologous to the function $\sigma(z)$, defined in the fourth lecture, so that the $\wp$-function may be expressed in terms of the theta functions, and consequently also the functions, *sn*, *cn* and *dn* may be expressed as in Art. 24 of my second lecture.

(*a*) *The four Weierstrassian sigma functions.* In Art. 34 of the fourth lecture, the Weierstrassian sigma function $\sigma(z)$ was defined as

$$\sigma(z)=z\Pi'\left\{\left(1-\frac{z}{\Omega_{m,n}}\right)e^{\frac{z}{\Omega_{m,n}}+\frac{1}{2}\left(\frac{z}{\Omega_{m,n}}\right)^2}\right\}$$

and in Art. 35 was given the relation

$$\wp(z_1)-\wp(z_2)=-\frac{\sigma(z_1+z_2)\sigma(z_1-z_2)}{\sigma^2(z_1)\sigma^2(z_2)}, \qquad \dots \quad (1)$$

Now let $\sigma_1(z)$, $\sigma_2(z)$, $\sigma_3(z)$ stand respectively for

$$e^{\frac{1}{2}\eta_1 z}\cdot\frac{\sigma(\omega-z)}{\sigma(\omega)},\quad e^{\frac{1}{2}(\eta_1+\eta_2)z}\,\frac{\sigma(\omega+\omega'-z)}{\sigma(\omega+\omega')},\quad e^{\frac{1}{2}\eta_2 z}\,\frac{\sigma(\omega'-z)}{\sigma(\omega')}.$$

Then, since

$$\wp(\omega)=e_1,$$

(1) gives

$$\wp(z)-e_1=-\frac{\sigma(z+\omega)\sigma(z-\omega)}{\sigma^2(z)\sigma^2(\omega)}=\left\{\frac{\sigma_1(z)}{\sigma(z)}\right\}^2,$$

because, as stated on p. 33,

$$\sigma(z+\Omega_{m,n})=\epsilon\sigma(z)\times e^{\eta\left(z+\frac{\Omega_{m,n}}{2}\right)},$$

so that

$$\sigma(z+\omega)=-\sigma(z-\omega)\times e^{\eta_1 z}$$

$$=\sigma(\omega-z)e^{\eta_1 z}.$$

Similarly, it can be proved that

$$\wp(z)-e_2=\left\{\frac{\sigma_2(z)}{\sigma(z)}\right\}^2,\quad \wp(z)-e_3=\left\{\frac{\sigma_3(z)}{\sigma(z)}\right\}^2.$$

(*b*) *Expression of the theta functions in terms of the sigma functions.* It can be easily proved that every integral function $f(z)$ of z with the period ω admits of being expanded in a convergent power series of the form

$$\sum_{n=-\infty}^{\infty} A_n \left(\frac{e^{2\pi i z}}{\omega}\right)^n.$$

Now, consider the function

$$f(z)\equiv e^{-\frac{\eta_1 z^2}{4\omega}+\frac{\pi i z}{2\omega}}\sigma(z).$$

It is obviously an integral function as $\sigma(z)$ is an integral function; further

$$f(z+2\omega)\equiv-\sigma(z)e^{-\frac{\eta_1(z+2\omega)^2}{4\omega}+\frac{\pi i(z+2\omega)}{2\omega}+\eta_1(z+\omega)}$$

$$=\sigma(z)e^{\frac{-\eta_1 z^2}{4\omega}+\frac{\pi i z}{2\omega}}=f(z) \qquad \dots \quad (2)$$

Thus $f(z)$ is an integral function with the period 2ω. Therefore, according to the aforesaid theorem,

$$f(z)=\sum_{-\infty}^{\infty} A_n u^{2n} \qquad \dots \quad (3)$$

where

$$u=e^{\frac{\pi i z}{2\omega}}.$$

But, in the same manner as (2) has been proved, it can be proved that

$$f(z+2\omega')=-u^{-2}f(z).$$

Hence, using (3), we have

$$-\sum A_{n+1} u^{2n} = f(z+2\omega') = \sum A_n u^{2n} q^{2n},$$

where $$q = e^{i\pi\frac{\omega'}{\omega}}.$$

Therefore, comparing the co-efficients,

$$A_{n+1} = -q^{2n} A_n,$$

i.e., $$(-1)^{n+1} q^{-(n+\frac{1}{2})^2} A_{n+1} = (-1)^n q^{-(n-\frac{1}{2})^2} A_n.$$

Thus $(-1)^n q^{-(n+\frac{1}{2})^2} A_n$ is a constant, say i C.

Therefore

$$f(z) = i\mathrm{C} \sum_{-\infty}^{\infty} (-1)^n q^{(n-\frac{1}{2})^2} u^{2n}.$$

But

$$i \sum_{-\infty}^{\infty} (-1)^n q^{(n-\frac{1}{2})^2} u^{2n-1} = \theta_1(v)^*$$

where $u = e^{\pi i v}$.

Thus it is proved that

$$\mathrm{C}\theta_1(v) = \sigma(z) e^{\frac{-\eta_1 z^2}{4\omega}},$$

where the constant C is easily seen to be $\dfrac{2\omega}{\theta_1'(0)}$.

Thus it is established that

$$\theta_1(v) \equiv 2\{q^{\frac{1}{4}} \sin \pi v - q^{\frac{9}{4}} \sin 3\pi v + q^{\frac{25}{4}} \sin 5\pi v - \ldots\}$$

$$= \frac{\theta_1'(0)}{2\omega} e^{\frac{-\eta_1 z^2}{4\omega}} . \sigma(z),$$

where $z = 2\omega v$.

---

* In the notation of Jacobi, used in Art. 25, this will be $\theta_1(\pi v)$. Similarly for $\theta_2, \theta_3, \theta_0$.

Similarly, it can be proved that

$$\theta_2(v) \equiv 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi v = \theta_2(0)e^{-\frac{\eta_1 z^2}{4\omega}}\sigma_1(z),$$

$$\theta_3(v) \equiv 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi v = \theta_3(0)e^{-\frac{\eta_1 z^2}{4\omega}}\sigma_2(z),$$

$$\theta_0(v) \equiv 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi v = \theta_0(0)e^{-\frac{\eta_1 z^2}{4\omega}}\sigma_3(z).$$

(c) *Expressions for the $\wp$-function in terms of the theta functions.*

Using the above formulae and those given in the end of (*a*), we have

$$\sqrt{\wp(z)-e_1} = \frac{1}{2\omega}\cdot\frac{\theta_1'(0)}{\theta_2(0)}\,\frac{\theta_2(v)}{\theta_1(v)}.$$

$$\sqrt{\wp(z)-e_2} = \frac{1}{2\omega}\cdot\frac{\theta_1'(0)}{\theta_3(0)}\cdot\frac{\theta_3(v)}{\theta_1(v)},$$

$$\sqrt{\wp(z)-e_3} = \frac{1}{2\omega}\cdot\frac{\theta_1'(0)}{\theta_0(0)}\,\frac{\theta_0(v)}{\theta_1(v)}.$$

2. *Mutual relations between the theta functions.* The following relations between the theta functions are easily deduced from the definitions :—

(I) $\theta_0(v+\frac{1}{2})=\theta_3(v),\ \theta_1(v+\frac{1}{2})=\theta_2(v),\theta_2(v+\frac{1}{2})=-\theta_1(v),$

$\theta_3(v+\frac{1}{2})=\theta_0(v).$

(II) $\theta_0(v+1)=\theta_0(v),\ \theta_1(v+1)=-\theta_1(v),\ \theta_2(v+1)=-\theta_2(v),$

$\theta_3(v+1)=\theta_3(v),$

(III) $\theta_0(v+\frac{1}{2}\tau)=iq^{-\frac{1}{4}}e^{-v\pi i}\theta_1(v),\ \theta_1(v+\frac{1}{2}\tau)=iq^{-\frac{1}{4}}e^{-v\pi i}\theta_0(v),$

$\theta_2(v+\frac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-v\pi i}\theta_3(v),\ \theta_3(v+\frac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-v\pi i}\theta_2(v).$

(IV) $\theta_0(v+\tau)=-q^{-1}e^{-2v\pi i}\,\theta_0(v),\ \theta_1(v+\tau)=-q^{-1}e^{-2v\pi i}\,\theta_1(v),$

$$\theta_2(v+\tau)=q^{-1}e^{-2v\pi i}\,\theta_2(v),\ \theta_3(v+\tau)=q^{-1}e^{-2v\pi i}\,\theta_3(v).$$

(V) $\theta_0(v+\tfrac{1}{2}+\tfrac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-v\pi i}\,\theta_2(v),\ \theta_1(v+\tfrac{1}{2}+\tfrac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-v\pi i}\,\theta_3(v),$

$$\theta_2(v+\tfrac{1}{2}+\tfrac{1}{2}\tau)=-iq^{-\frac{1}{4}}e^{-v\pi i}\theta_0(v),\theta_3(v+\tfrac{1}{2}+\tfrac{1}{2}\tau)=iq^{-\frac{1}{4}}e^{-v\pi i}\theta_1(v).$$

(VI) $\theta_0(v+1+\tau)=-q^{-1}e^{-2v\pi i}\theta_0(v),\ \theta_1\ (v+1+\tau)=q^{-1}e^{-2v\pi i}\theta_1(v),$

$$\theta_2(v+1+\tau)=-q^{-1}e^{-2v\pi i}\,\theta_2(v),\ \theta_3(v+1+\tau)=q^{-1}e^{-2v\pi i}\,\theta_3(v).$$

(VII) $\theta_0(v+m+n\tau)=(-1)^n\,q^{-n^2}\,e^{-2nv\pi i}\theta_0(v),$

$$\theta_1(v+m+n\tau)=(-1)^{m+n}\,q^{-n^2}\,e^{-2nv\pi i}\,\theta_1(v),$$

$$\theta_2(v+m+n\tau)=(-1)^{m}\,q^{-n^2}\,e^{-2nv\pi i}\theta_2(v),$$

$$\theta_3(v+m+n\tau)=q^{-n^2}\,e^{-2nv\pi i}\,\theta_3(v)\ ;$$

m and n being any integers.

3. *The zeroes of the theta functions.* It is obvious from the expression for $\theta_1\ (v)$ in terms of $\sigma\ (z)$ that the zeroes of the two are the same. But $\sigma\ (z)$ vanishes, when

$$z=\Omega_{m,n}=2m\omega+2n\omega'\ ,$$

m and n being any integers from $-\infty$ to ∞.

Therefore $\theta_1\ (v)$ vanishes when

$$2\ \omega v=2m\omega+2n\omega',$$

$$i.e.,\qquad v=m+n\tau.$$

The zeroes of the other theta functions are obtained by using the above result and the relations given in the preceding article.

For example,

$$\theta_2(v+\tfrac{1}{2})=-\theta_1(v)\ ;$$

therefore $\theta_2(v+\tfrac{1}{2})$ vanishes when $v=m+n\tau$, *i.e.*, $\theta_2(v)$ vanishes when $v=m+n\tau+\tfrac{1}{2}$.

The zeroes of the various theta functions are given in tabular form as follows :

Function	$v=$	$u^2=e^{2\pi i v}=$
$_1$	$m+n\tau$	q^{2n}
$_2$	$m+n\tau+\frac{1}{2}$	$-q^{2n}$
$_3$	$m+n\tau+\frac{1}{2}+\frac{\tau}{2}$	$-q^{2n+1}$
θ_0	$m+n\tau+\frac{\tau}{2}$	q^{2n+1}

4. *Infinite products for the theta functions.* Consider first $\theta_3(v)$. Its zeroes are given by

$$u^2=-q^{-1}\ ,\ -q^{-3}\ ,\ \ldots\ldots\ ,$$

$$u^2=-q\quad ,\ -q^3\ ,\ \ldots\ldots$$

Therefore, by Weierstrass's factor-theorem,

$$F(v)\equiv\prod_1^\infty \{(1+q^{2n-1}u^2)(1+q^{2n-1}u^{-2})\}$$

is an integral-function of v having the same zeroes as $\theta_3\ (v)$.

Also it is obvious from the above definition of F (v) that

$$F(v+1)=F(v),\ F(v+\tau)=q^{-1}u^{-2}F(v).$$

But, from the relations (II) and (IV) of Art. 2, it is clear that $\theta_3\ (v)$ behaves in the same manner. Therefore $\frac{\theta_3(v)}{F(v)}$ is a doubly periodic function of v with the periods 1 and τ but has no pole. Therefore it must be constant, say C. Thus it is proved that

$$\theta_3(v)=C.\prod_1^\infty (1+q^{2n-1}\,u^2)(1+q^{2n-1}\,u^{-2})$$

$$=C.\prod_1^\infty (1+2q^{2n-1}\cos 2\pi v+q^{4n-2}).$$

Consider $\theta_0(v)$. According to (I) of Art. 2,

$$\theta_0(v)=\theta_3(v+\tfrac{1}{2})=C\prod_1^\infty(1-q^{2n-1}u^2)(1-q^{2n-1}u^{-2})$$

$$=C\prod_1^\infty(1-2q^{2n-1}\cos 2\pi v+q^{4n-2}).$$

Similarly,

$$\theta_2(v)=C.q^{\frac{1}{4}}(u+u^{-1})\prod_1^\infty(1+q^{2n}u^2)(1+q^{2n}u^{-2})$$

$$=2C.\ q^{\frac{1}{4}}\cos\pi v\prod_1^\infty(1+2q^{2n}\cos 2\pi v+q^{4n});$$

$$\theta_1(v)=Cq^{\frac{1}{4}}\left(\frac{u-u^{-1}}{i}\right)\prod_1^\infty(1-q^{2n}u^2)(1-q^{2n}u^{-2})$$

$$=2Cq^{\frac{1}{4}}\sin\pi v\prod_1^\infty(1-2q^{2n}\cos 2\pi v+q^{4n}).$$

These results agree with those given under (*b*) on page 22.

5. *Expressions for the quantities e_1, e_2, e_3, k, K in terms of the null values of the theta functions.* (*a*) I will first prove that

$$\theta'_1(o)=\pi\theta_2(o)\theta_3(o)\theta_0(o) \qquad \text{(A)}$$

and then deduce that

$$\sqrt{e_1-e_2}=\frac{\pi}{2\omega}\theta_0{}^2(o),\ \sqrt{e_1-e_3}=\frac{\pi}{2\omega}\theta_3{}^2(o),\ \sqrt{e_2-e_3}=\frac{\pi}{2\omega}\theta_2{}^2(o).$$

Proof of (A) :—

From (*c*) of Art. 1,

$$\sqrt{\wp(2\omega v)-e_1}=\frac{1}{2\omega}\frac{\theta'_1(o)}{\theta_2(o)}\cdot\frac{\theta_2(v)}{\theta_1(v)}\cdot \qquad \dots \qquad \dots \quad (1)$$

Now expanding $\dfrac{\theta_2(v)}{\theta_2(o)}$ by Maclaurin's theorem and remembering that it is an even function, we get

$$\frac{\theta_2(v)}{\theta_2(o)}=1+\frac{\theta''_2(o)}{\theta_2(o)}\cdot\frac{v^2}{2!}+\dots$$

Similarly,

$$\frac{\theta_1(v)}{\theta'_1(o)}=v+\frac{\theta'''_1(o)}{\theta'_1(o)}\cdot\frac{v^3}{3!}+\ldots$$

Substituting these values in (1), we have

$$\sqrt{\wp(2\omega v)-e_1}=\frac{1}{2\omega v}\left[1+\left\{\frac{\theta''_2(o)}{\theta_2(o)}-\frac{1}{3}\,\frac{\theta'''_1(o)}{\theta'_1(o)}\right\}\frac{v^2}{2}+\ldots\right].$$

Squaring both the sides and noting that the expansion of $\wp(2\omega v)$ does not contain any term independent of v, we have by equating the constant terms

$$e_1=\frac{1}{4\omega^2}\left\{\frac{1}{3}\,\frac{\theta'''_1(o)}{\theta'_1(o)}-\frac{\theta''_2(o)}{\theta_2(o)}\right\}.$$

Similarly

$$e_2=\frac{1}{4\omega^2}\left\{\frac{1}{3}\frac{\theta_1'''(0)}{\theta_1'(0)}-\frac{\theta_3''(0)}{\theta_3(0)}\right\},$$

$$e_3=\frac{1}{4\omega^2}\left\{\frac{1}{3}\frac{\theta_1'''(0)}{\theta_1'(0)}-\frac{\theta_0''(0)}{\theta_0(0)}\right\}.$$

But $e_1+e_2+e_3=0.$

Therefore

$$\frac{\theta_1'''(0)}{\theta_1'(0)}=\frac{\theta''_2(0)}{\theta_2(0)}+\frac{\theta_3''(0)}{\theta_3(0)}+\frac{\theta_0''(0)}{\theta_0(0)}. \qquad \ldots \quad (2)$$

Now each θ satisfies

$$\frac{\partial^2\theta(v)}{\partial v^2}=4i\pi\frac{\partial\theta(v)}{\partial\tau},$$

$$\frac{\partial^3\theta(v)}{\partial v^3}=4i\pi\,\frac{\partial^2\theta(v)}{\partial v\,\partial\tau}$$

Therefore, putting $v=0$ in the above and using the result in (2), we have

$$\frac{1}{\theta_1'(0)}\frac{\partial}{\partial\tau}.\theta'_1(0)=\frac{1}{\theta_2(0)}\frac{\partial}{\partial\tau}\theta_2(0)$$

$$+\frac{1}{\theta_3(0)}\frac{\partial}{\partial\tau}\theta_3(0)+\frac{1}{\theta_0(0)}\frac{\partial}{\partial\tau}\theta_0(0);$$

whence, by integrating both the sides,

$$\theta_1'(0)=c\theta_2(0)\theta_3(0)\theta_0(0),$$

c being a constant.

By making the substitution $v=0$ in the series for the θ's and comparing the co-efficients of $q^{\frac{1}{4}}$ we have $c=\pi$.

Thus the result (A) is established.

(*b*) Since $\wp(\omega)=e_1$, putting $v=\frac{1}{2}$ in

$$\sqrt{\wp(2\omega v)-e_3}=\frac{1}{2\omega}\frac{\theta_1'(0)}{\theta_0(0)}\cdot\frac{\theta_0(v)}{\theta_1(v)},$$

we have

$$\sqrt{e_1-e_3}=\frac{1}{2\omega}\frac{\theta_1'(0)}{\theta_0(0)}\cdot\frac{\theta_0(\frac{1}{2})}{\theta_1(\frac{1}{2})}$$

$$=\frac{1}{2\omega}\frac{\theta_1'(0)}{\theta_0(0)}\cdot\frac{\theta_3(0)}{\theta_2(0)},\quad \text{by (I) of Art 2.}$$

Thus $\sqrt{e_1-e_3}=\frac{\pi}{2\omega}\theta_3^2(0)$, by (A).

Similarly,

$$\sqrt{e_1-e_2}=\frac{\pi}{2\omega}\theta_0^2(0),\quad \sqrt{e_2-e_3}=\frac{\pi}{2\omega}\theta_2^2(0).$$

(*c*) As proved in Art. 23 of the second lecture,

$$\sqrt{\wp(z)-e_3}=\frac{\sqrt{e_1-e_3}}{\text{sn}(\sqrt{e_1-e_3}\,z)},$$

k being $\sqrt{\dfrac{e_2-e_3}{e_1-e_3}}$.

Therefore, using the results in the end of (*b*), we have

$$k=\frac{\theta_2^2(0)}{\theta_3^2(0)}.$$

Also $k'=\sqrt{\dfrac{e_1-e_2}{e_1-e_3}}=\dfrac{\theta_0^2(0)}{\theta_3^2(0)}$.

(*d*) Since

$$\mathrm{sn}(\mathbf{K})=\mathbf{1},$$

it follows from

$$\sqrt{\wp(z)-e_3}=\frac{\sqrt{e_1-e_3}}{\mathrm{sn}(\sqrt{e_1-e_3}\ z)},$$

that

$$\omega\sqrt{e_1-e_3}=\mathrm{K},$$

i.e.,

$$\mathrm{K}=\frac{\pi}{2}\theta^2_3(0).$$

Similarly,

$$\mathrm{K}'=\omega' i\sqrt{e_1-e_3}=\frac{\pi}{2}\,i\tau\theta_3^2(0).$$

(*e*) Hence

$$\tau=-i\frac{\mathrm{K}'}{\mathrm{K}}$$

and, consequently,

$$q=e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}}.$$

6. *Expressions for the sn, cn, dn functions in terms of the theta functions.*

From

$$\sqrt{\wp(z)-e_3}=\frac{1}{2\omega}\,\frac{\theta_1'(0)}{\theta_0(0)}\cdot\frac{\theta_0(v)}{\theta_1(v)},$$

we have, by using the expression for $\operatorname{sn}(\sqrt{e_1-e_3}\,z)$ given in Art. 23 of the second lecture,

$$\operatorname{sn}(\sqrt{e_1-e_3}\,z)=\sqrt{e_1-e_3}\,.\,2\omega\,.\,\frac{\theta_0(0)}{\theta_1'(0)}\,.\,\frac{\theta_1(v)}{\theta_0(v)},$$

i.e.,

$$\operatorname{sn}(2\mathrm{K}v)=\frac{1}{\sqrt{k}}\,\frac{\theta_1(v)}{\theta_0(v)},$$

by using (A) and the expression for k given in (*c*) of Art. 5.

Also

$$\operatorname{cn}(\sqrt{e_1-e_3}\,z)=\frac{\sqrt{\wp(z)-e_1}}{\sqrt{\wp(z)-e_3}}.$$

Therefore, substituting the expressions from (*c*) of Art. 2, we have

$$\operatorname{cn}(\sqrt{e_1-e_3}\,z)=\frac{\theta_2(v)}{\theta_0(v)},$$

$$i.e.,\quad \operatorname{cn}(2\mathrm{K}v)=\frac{\theta_0(0)}{\theta_2(0)}\,\frac{\theta_2(v)}{\theta_0(v)}=\sqrt{\frac{k'}{k}}\,.\,\frac{\theta_2(v)}{\theta_0(v)}.$$

Similarly,

$$\operatorname{dn}(2\mathrm{K}v)=\sqrt{k'}\,\frac{\theta_3(v)}{\theta_0(v)}.$$

7. *Determination of q for given k ; Hermite's functions.*

(*a*) From the results

$$\sqrt{k}=\frac{\theta_2(0)}{\theta_3(0)},\quad \sqrt{k'}=\frac{\theta_0(0)}{\theta_3(0)},$$

are obtained, by the inversion of the series, respectively

$$q=\frac{1}{2^4}\,k^2+\frac{1}{2^5}\,k^4+\frac{21}{2^{10}}\,k^6+\frac{31}{2^{11}}\,k^8+\frac{6257}{2^{19}}\,k^{10}+\ldots$$

and

$$q=\frac{1}{2}\,l+2\left(\frac{1}{2}\,l\right)^5+15\left(\frac{1}{2}\,l\right)^9+150\left(\frac{1}{2}\,l\right)^{13}+\ldots,$$

l standing for $\dfrac{1-\sqrt{k'}}{1+\sqrt{k'}}$.

The latter series is so rapidly convergent that, if $|l| \not> \frac{1}{2}$, the error in taking the first two or three terms is numerically less than

$$\frac{1}{30}\, l^9 \text{ or } \frac{1}{50}\, l^{13}.$$

It should be noticed that, although for a given k or k', there are an infinite number of values for q, one value is obtained by means of the series given above ; the other values are deduced by the linear transformation of theta functions.

(*b*) Let

$$\frac{iK'}{K} = -\tau$$

be denoted by t, then $k^{\frac{1}{4}}$ and $k'^{\frac{1}{4}}$, considered as functions of t, are respectively Hermite's functions $\phi(t)$ and $\psi\ (t)$.

The following results relating to ϕ and ψ are obvious :

$$\phi^8(t) + \psi^8(t) = 1,$$

$$\phi\left(-\frac{1}{t}\right) = \psi(t),$$

$$\phi\ \ (t+1) = e^{\frac{i\pi}{8}}\, \frac{\phi(t)}{\psi(t)},$$

$$\psi(t+1) = \frac{1}{\psi(t)}.$$

It can be also proved that $\phi\left(\dfrac{c+dt}{a+bt}\right)$ and $\psi\left(\dfrac{c+dt}{a+bt}\right)$ admit of being expressed simply in terms of $\phi(t)$ and $\psi(t)$; a, b, c, d being integers subject to the condition

$$ad - bc = 1.$$

8. *Jacobi's fundamental theorem.* If

$$w' = \tfrac{1}{2}\ (w+x+y+z),\ x' = \tfrac{1}{2}\ (w+x-y-z),$$

$$y' = \tfrac{1}{2}(w-x+y-z),\ z' = \tfrac{1}{2}\ (w-x-y+z),$$

then

$$\theta_3(w)\theta_3(x)\theta_3(y)\theta_3(z)+\theta_2(w)\theta_2(x)\theta_2(y)\theta_2(z)$$

$$=\theta_3(w')\theta_3(x')\theta_3(y')\theta_3(z')+\theta_2(w')\theta_2(x',\theta_2(y')\theta_2(z').$$

Proof:

$$\theta_3(x)=\sum_{-\infty}^{\infty} q^{n^2} e^{2n\pi xi}=\sum_{-\infty}^{\infty} e^{n^2 \log q+2n\pi xi}$$

$$=e^{\frac{\pi^2x^2}{\log q}} \sum_{-\infty}^{\infty} e^{\left(\frac{2n}{2}\log\ q+\pi xi\right)^2/\log q}.$$

Similarly,

$$\theta_2(x)=e^{\frac{\pi^2x^2}{\log q}} \sum_{-\infty}^{\infty} e^{\left(\frac{2n+1}{2}\log q+\pi xi\right)^2/\log q}$$

Therefore

$$\theta_3(w)\theta_3(x)\theta_3(y)\theta_3(z)=\{e^{\pi^2(w^2+x^2+y^2+z^2)/\log q}\}$$

$$\times\sum_{n}\sum_{n'}\sum_{n''}\sum_{n'''} e^{S/\log q},$$

$$\theta_2(w)\theta_2(x)\theta_2(y)\theta_2(z)=\{e^{\pi^2(w^2+x^2+y^2+z^2)/\log q}\}$$

$$\times\sum_{n}\sum_{n'}\sum_{n''}\sum_{n'''} e^{T/\log q},$$

where

$$S=\left(\frac{2n}{2}\log\ q+w\pi i\right)^2+\left(\frac{2n'}{2}\log q+x\pi i\right)^2+\left(\frac{2n''}{2}\log q+y\pi i\right)^2$$

$$+\left(\frac{2n'''}{2}\log q+z\pi i\right)^2$$

and

$$T=\left(\frac{2n+1}{2}\log q+w\pi i\right)^2+\left(\frac{2n'+1}{2}\log\ q+x\pi i\right)^2$$

$$+\left(\frac{2n''+1}{2}\log q+y\pi i\right)^2+\left(\frac{2n'''+1}{2}\log q+z\pi i\right)^2,$$

n, n', n'', n''' being integers all independent of one another and taking all the values from $-\infty$ to ∞.

As the factors of $\frac{1}{2}$ log q in S are all even numbers and its factors in T are all odd numbers, it follows that

$$\theta_3(w)\theta_3(x)\theta_3(y)\theta_3(z)+\theta_2(w)\theta_2(x)\theta_2(y)\theta_2(z)$$
$$=\{e^{\pi^2(w_1^2+x^2+y^2+z^2)/\log q}\}\times\Sigma' e^{H/\log q},$$

where

$$H=\left(\frac{a}{2}\log q+w\pi i\right)^2+\left(\frac{b}{2}\log q+x\pi i\right)^2+\left(\frac{c}{2}\log q+y\pi i\right)^2$$
$$+\left(\frac{d}{2}\log q+z\pi i\right)^2,$$

the sign Σ' denoting that the summation is to extend to all possible integers a, b, c, d, which are together even or together odd.

Now it is not difficult to see that $\Sigma' e^{H/\log q}$ remains unaltered if

$$H=\left(\frac{a'}{2}\log q+w'\pi i\right)^2+\left(\frac{b'}{2}\log q+x'\pi i\right)^2$$
$$+\left(\frac{c'}{2}\log q+y'\pi i\right)^2+\left(\frac{d'}{2}\log q+z'\pi i\right)^2,$$

where

$$a'=\tfrac{1}{2}(a+b+c+d),\ b'=\tfrac{1}{2}(a+b-c-d),$$
$$c'=\tfrac{1}{2}(a-b+c-d),\ d'=\tfrac{1}{2}(a-b-c+d);$$

for

$$w'^2+x'^2+y'^2+z'^2=w^2+x^2+y^2+z^2,$$

and a', b', c', d' run through all the possible systems of four even or four odd integers just as a, b, c, d, do.

Therefore

$$\theta_3(w)\theta_3(x)\theta_3(y)\theta_3(z)+\theta_2(w)\theta_2(x)\theta_2(y)\theta_2(z)$$
$$=e^{\pi^2\{z'^2+y'^2+x'^2+w'^2\}/\log q}\times\Sigma' e^{H/\log q}$$

which equals

$$\theta_3(w')\theta_3(x')\theta_3(y')\theta_3(z')+\theta_2(w')\theta_2(x')\theta_2(y')\theta_2(z').$$

9. *Addition theorems for the theta functions.* There are altogether 256 addition theorems which can be all deduced from the fundamental theorem. By giving special values to the variables, 36 formulæ can be deduced for

$$\theta_\alpha(v+v_1)\theta_\alpha(v-v_1) \text{ and } \theta_\alpha(v+v_1)\,\theta_\beta(v-v_1), \ (\alpha,\beta=0, 1, 2, 3).$$

Of these the following may be mentioned, c_0. c_2, c_3 standing $\theta_0(0)$, $\theta_2(0)$, $\theta_3(0)$ respectively :—

$$\text{(VIII)} \quad c_0{}^2\theta_1(v+v_1)\theta_1(v-v_1)=\theta_1{}^2(v)\theta_0{}^2(v_1)-\theta_0{}^2(v)\theta_1{}^2(v_1)$$
$$=\theta_3{}^2(v)\theta_2{}^2(v_1)-\theta_2{}^2(v)\theta_3{}^2(v_1)$$

$$\text{(IX)} \quad c_3{}^2\theta_1(v+v_1)\theta_1(v-v_1)=\theta_1{}^2(v)\theta_2{}^2(v_1)-\theta_2{}^2(v)\theta_1{}^2(v_1)$$
$$=\theta_0{}^2(v)\theta_3{}^2(v_1)-\theta_3{}^2(v)\theta_0{}^2(v_1).$$

Putting $v_1=0$, we have

$$\text{(X)} \quad c_3{}^2\,\theta_0{}^2(v)=c_0{}^2\theta_3{}^2(v)+c_2{}^2\theta_1{}^2(v),$$

$$\text{(XI)} \quad c_3{}^2\,\theta_1{}^2(v)=c_2\,\theta_0{}^2(v)-c_0{}^2\theta_2{}^2(v),$$

$$\text{(XII)} \quad c_3{}^2\theta_2{}^2(v)=c_2{}^2\theta_3{}^2(v)-c_0{}^2\theta_1{}^2(v)$$

$$\text{(XIII)} \quad c_3{}^2\theta_3{}^2(v)=c_0{}^2\theta_0{}^2(v)+c_2{}^2\theta_2{}^2(v).$$

Hence, putting $v=0$, we have

$$\text{(XIV)} \quad c_3{}^4=c_0{}^4+c_2{}^4.$$

$$\text{(XV)} \quad c_0{}^8+c_2{}^8+c_3{}^8=2(c_2{}^4c_3{}^4+c_0{}^4c_3{}^4-c_0{}^4c_2{}^4)$$

$$\text{(XVI)} \quad (c_0{}^8+c_2{}^8+c_3{}^8)^2=2(c_0{}^{16}+c_2{}^{16}+c_3{}^{16})$$
$$=4(c_2{}^8c_3{}^8+c_0{}^8c_3{}^8+c_0{}^8c_2{}^8)$$

10. *Theta functions of p variables.* (a) By the theta function of p variables v_1, v_2,...v_p and the given parameters $a_{kl}=a_{lk}$ is understood the function

$$\theta(v_1, v_2,...v_p)\equiv \sum_{n_1, n_2...n_p} e^{\pi i\Omega} \qquad ... \quad (1)$$

where

$$\Omega=\chi(n)+2\sum_{\mu=1}^{p} n_\mu\, v_\mu\,,$$

$\chi(n)$ standing for the quadratic form

$$\sum_{k\,l=1}^{p} n_k n_l a_{kl} = (a_{11}n_1^2 + 2a_{12}n_1n_2 + \ldots + a_{pp}n_p^2)$$

of the p variables and the summation $\sum\limits_{n_1,\,n_2 \ldots n_p}$ extending over all integral values of $n_1, n_2, \ldots n_p$ from $-\infty$ to ∞ taken independently of one another.

The $\frac{1}{2}p(p+1)$ constants, a's, which are arbitrary except for certain conditions for the convergence of the series (**1**), are called the *moduli* or *parameters* of $\theta(v_1, v_2, \ldots v_p)$, frequently denoted for brevity by $\theta(v)$.

If a'_{kl} denote the imaginary part of a_{kl}, then the series (**1**) is convergent for all finite values of the p variables if, and only if, the quadratic form $\sum\limits_{k\,l=1}^{p} n_k n_l a'_{kl}$ is definite and positive; hence the conditions, referred to above, for the a's. When these conditions are satisfied, the series is absolutely convergent and $\theta(v)$ represents a function which is always finite and continuous for all finite values of the variables.

(β) The function $\theta(v)$ has the following properties:

(*i*) It is an integral and transcendental function of the variables.

(*ii*) $\theta(v_1, \ldots\ldots, v_\mu + 1, \ldots v_p) = \theta(v_1, \ldots v_\mu, \ldots v_p)$.

(*iii*) $\theta(v_1 + a_{1l}, v_2 + a_{2l}, \ldots, v_p + a_{pl}) = \theta(v)e^{-\pi i a_{ll} - 2\pi i v_l}$.

(*iv*) $4\pi i \dfrac{\partial \theta}{\partial a_{kk}} = \dfrac{\partial^2 \theta}{\partial v_k^2}$, $\quad 2\pi i \dfrac{\partial \theta}{\partial a_{kl}} = \dfrac{\partial^2 \theta}{\partial v_k \partial v_l}$.

The second and third equations may be combined in the statement that for integral values of h's and g's,

$$\theta(v_1 + \omega_1, v_2 + \omega_2, \ldots, v_p + \omega_p) \equiv \theta(v + \omega) = \theta(v).e^{-\pi i \chi(g) - 2\pi i \Sigma g_l v_l},$$

where
$$\omega_k = h_k + \sum_{l=1}^{p} g_l a_{kl}.$$

By means of the first three properties the function is determined having a factor dependent on the a's; by the first four properties, it is determined having a factor independent even of the a's.

The $2p$ systems of p quantities,

$$\begin{array}{ll} 1, 0, \ldots 0, & a_{11}\; a_{21} \ldots a_{p1} \\ 0, 1, \ldots 0, & a_{12}\; a_{22} \ldots a_{p2} \\ \ldots\;\ldots\;\ldots & \ldots\;\ldots\;\ldots \\ 0, 0, \ldots 1, & a_{1p}\; a_{2p} \ldots a_{pp}, \end{array}$$

are each called a *connected period system* or briefly a *period* of the theta function.

11. *Characteristics*; *general theta function.* (α) If $c_1, c_2, \ldots c_p$ are any p quantities, then real quantities $h_1, h_2, \ldots h_p, g_1, g_2, \ldots g_p$ can be always so determined *uniquely* that

$$c_k = h_k + \sum_l g_l a_{kl}.$$

The system so determined is denoted by

$$\begin{bmatrix} g_1, g_2, \ldots g_p \\ h_1, h_2, \ldots h_p \end{bmatrix} \text{ or briefly } \begin{bmatrix} g \\ h \end{bmatrix}$$

and is called the *period characteristic of the given quantities* c_k.

(β) The general theta function is denoted by

$$\theta \begin{bmatrix} g_1, g_2, \ldots g_p \\ h_1, h_2, \ldots h_p \end{bmatrix} (v_1, v_2, \ldots v_p) \text{ or briefly } \theta \begin{bmatrix} g \\ h \end{bmatrix} (v)$$

and is equal to

$$\theta(v_1 + c_1, v_2 + c_2, \ldots v_p + c_p) e^{\pi i \phi},$$

where

$$\phi = \chi(g) + 2 \sum_k g_k (v_k + h_k).$$

The system $\begin{bmatrix} g \\ h \end{bmatrix}$ is called the *characteristic of this theta function.*

(γ) The general theta function has the following properties :—

(*i*) It is an integral transcendental function of $v_1, v_2, \ldots v_p$.

$$(ii)\quad \theta\begin{bmatrix} g \\ h \end{bmatrix}(v_1, \ldots, v_k+1, \ldots v_p) = e^{2g_k\pi i}\,\theta\begin{bmatrix} g \\ h \end{bmatrix}(v),$$

$$(iii)\quad \theta\begin{bmatrix} g \\ h \end{bmatrix}(v_1+a_{1k}, v_2+a_{2k}, \ldots, v_p+a_{pk})$$

$$=\theta\begin{bmatrix} g \\ h \end{bmatrix}(v).e^{-\pi i(a_{kk}+2v_k+2h_k)}$$

$$(iv)\quad 4\pi i\,\frac{\partial\theta}{\partial a_{kk}} = \frac{\partial^2\theta}{\partial v_k^2}, \quad 2\pi i\,\frac{\partial\theta}{\partial a_{kl}} = \frac{\partial^2\theta}{\partial v_k\,\partial v_l}.$$

The second and third equations may be combined in the statement that

$$\theta\begin{bmatrix} g \\ h \end{bmatrix}(v+\omega) = e^{\pi i\psi}\theta\begin{bmatrix} g \\ h \end{bmatrix}(v),$$

where, with integral g''s and h''s,

$$\omega_l = h'_l + \sum_k g'_k a_{lk},$$

ψ standing for

$$-\chi(g') - 2\sum g'_k v_k + 2\sum(h'_k g_k - h_k g'_k).$$

By the first three properties, the function is determined having a factor dependent upon the parameters a_{kl} ; by the first four properties it is determined having a factor independent of the a's.

(δ) To each characteristic there corresponds a theta function. The characteristic is called *even* or *odd* according as the corresponding function is even or odd.

The characteristic $\begin{bmatrix} g \\ h \end{bmatrix}$ is even or odd according as the sum $\sum g_k h_k$ is even or odd.

By the sum of two or more characteristics is understood that characteristic whose elements are congruent (mod. 2) with the sums of homologous elements in each of the given characteristics.

The theta function defined in Art. 10 corresponds to the characteristic 0 ; so that $g=0$, $h=0$.

• The number of even theta functions is $2^{p-1}(2^p+1)$ and the number of odd theta functions is $2^{p-1}(2^p-1)$.

Thus for $p=1$ there are three even and one odd theta functions; they are $\theta[{}^0_0](v)$, $\theta[{}^0_1](v)$, $\theta[{}^1_0](v)$, $\theta[{}^1_1](v)$ corresponding respectively to the functions $\theta_3(v)$, $\theta_0(v)$, $\theta_2(v)$, $\theta_1(v)$ of Art. 1.

For $p=2$ there are 10 even and 6 odd theta functions; they are

$$(1)\ \theta\begin{bmatrix}0,0\\0,0\end{bmatrix}(v);\ \theta\begin{bmatrix}0,0\\1,0\end{bmatrix}(v);\ \theta\begin{bmatrix}0,1\\0,0\end{bmatrix}(v),\ \theta\begin{bmatrix}1,0\\0,0\end{bmatrix}(v);$$

$$\theta\begin{bmatrix}1,0\\0,1\end{bmatrix}(v);\ \theta\begin{bmatrix}0,1\\1,0\end{bmatrix}(v);\ \theta\begin{bmatrix}1,1\\0,0\end{bmatrix}(v);\ \theta\begin{bmatrix}1,1\\1,1\end{bmatrix}(v);$$

$$\theta\begin{bmatrix}0,0\\0,1\end{bmatrix}(v);\ 0\ \begin{bmatrix}0,0\\1,1\end{bmatrix}(v)$$

all even and

$$(2)\ \theta\begin{bmatrix}0,1\\0,1\end{bmatrix}(v);\ \theta\begin{bmatrix}1,1\\0,1\end{bmatrix}(v);\ \theta\begin{bmatrix}1,0\\1,0\end{bmatrix}(v);\ \theta\begin{bmatrix}1,1\\1,0\end{bmatrix}(v);$$

$$\theta\begin{bmatrix}0,1\\1,1\end{bmatrix}(v);\ \theta\begin{bmatrix}1,0\\1,1\end{bmatrix}(v),$$

all odd*

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* The first few terms of some of these may be given here :

$$\theta\begin{bmatrix}0,0\\0,0\end{bmatrix}(v)=1+P\cos 2\pi v_1+Q\cos 2\pi v_1+\dots,$$

$$\theta\begin{bmatrix}0,0\\1,0\end{bmatrix}(v)=1-P\cos 2\pi v_1+Q\cos 2\pi v_2+\dots,$$

$$\theta\begin{bmatrix}1,0\\1,0\end{bmatrix}(v)=2\sqrt[4]{P}\sin\pi v_1+2Q\sqrt[4]{P}\left\{e^{\pi i a_{12}}\sin(\pi v_1+2\pi v_2)\right.$$
$$\left.+e^{-\pi a_{12}}\sin\begin{pmatrix}\pi v_1\\-2\pi v_2\end{pmatrix}\right\}+\dots,$$

$$\theta\begin{bmatrix}1,1\\1,0\end{bmatrix}(v)=2\sqrt[4]{PQ}\left\{e^{\frac{1}{4}\pi i a_{12}}\sin\pi(v_1+v_2)-e^{-\frac{1}{4}\pi i a_{12}}\right.$$
$$\left.\times\sin\pi(v_1-v_2)+\dots\right\}.$$

where $P=e^{\pi i a_{11}}$, $Q=e^{\pi i a_{22}}$.

12. *Klein's sigma functions for $p=2$.* Let

$$\sigma(v_1, v_2)=\frac{1}{C}\,\theta\,(v_1, v_2)e-\chi, \tag{1}$$

where

$$\chi=\frac{1}{20}\left(\sum_1^{10}\frac{\theta_{11}}{\theta}\,v_1^2+2\sum_1^{10}\frac{\theta_{12}}{\theta}\,v_1v_2+\sum_1^{10}\frac{\theta_{22}}{\theta}\,v_2^2\right),$$

the sum extending over the 10 even theta functions, θ denoting the value of $\theta(v_1, v_2)$, θ_μ that of $\frac{\partial\theta(v_1, v_2)}{\partial v_\mu}$ and $\theta_{\mu\nu}$ that of $\frac{\partial^2\theta(v_1, v_2)}{\partial v_\mu \partial v}$ all for $v_1=v_2=0$, and C being a constant which equals in the case of an even theta function its value θ for $v_1=v_2=0$ and in the case of an odd theta function $\frac{1}{p_1}\theta_1$ or $\frac{1}{p_2}\theta_2$, p_1, p_2 being suitable constants. The functions σ were introduced by Klein† and are called Klein's *hyper elliptic sigma functions* for $p=2$. They are 16 in number, 10 even and 6 odd.

These functions possess this advantage over the theta functions that, for any linear transformation of periods, one sigma function simply changes into another, whereas for the same transformation a theta function changes into another theta function multiplied by an exponential factor. The sigma functions of Klein are generalizations of the four sigma functions of Weierstrass for the case of a single variable. The functions $\sigma(v_1, v_2)$ have been expanded in powers of v_1 and v_2.

13. *Representation of an Abelian function by means of theta or sigma functions.*

In Art. 68 of the sixth lecture I gave the representation of a hyper-elliptic function x by means of the theta functions of two variables, w_1, w_2, where

$$w_1=\int\frac{dx}{s}\,,\ w_2=\int\frac{xdx}{s}$$

and $$s=\sqrt{(x-c_1)(x-c_2)(x-c_3)(x-c_4)(x-c_5)(x-c_6)}.$$

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† In lectures delivered at the University of Göttingen from Summer 1887 to Summer 1889.

• Generally, every single-valued function of p variables and $2p$ periods having no essential singularity in the finite domain can be expressed rationally by means of suitably chosen theta functions.

For example, if

$$\theta(v_1-e_1,\ v_2-e_2,\dots v_p-e_p)$$

be denoted by $\theta(v-e)$, then the expression

$$\frac{\theta(v-A')\theta(v-A^2)\dots\theta(v-A^p)}{\theta(v-C^1)\theta(v-C^2)\dots\theta(v-C^p)}\cdot e^{-2\sum\limits_{k=1}^{p}\mu_k v_k} \tag{2}$$

is an Abelian function of the variables $v_1, v_2,\dots v_p$; the constants $A', A^2,\dots A^p, C^1, C^2,\dots C^p$ being subject only to the condition that none of the theta functions in (2) identically vanishes for every point x, the μ's being constants and x being given by

$$\int_a^x dv_k = v_k,\ (k=1, 2,\dots p).$$

The most general function of the co-ordinates of x admits of being expressed in terms of $v_1, v_2,\dots v_p$ in the form (2).

APPENDIX C.

Notes, additions and corrections.

Page 1. The quotation is from Dr. Einar Hille's translation in the *Annals of Mathematics*, Vol. 21, of Mittag-Leffler's book, " En Metod att Komma I Analytisk besittning af de Elliptiska Funktionerna " (Helsingfors, 1876).

Page 2. The problem of dividing the quadrant of a lemniscate into n equal parts engaged the attention of many eminent mathematicians after Fagnano. Gauss and Abel proved that the division of the lemniscate by means of ruler and compass is always possible when such a division of the circle is possible. Thus, if n is equal to $2^l p_1 p_2 \ldots$, where each p is a distinct prime of the form $2^{2^t}+1$, l and t being integers, the quadrant of the lemniscate can be divided by ruler and compass into n equal parts. See Abel's " Oeuvres," t. 1, p. 361.

Page 10. Line 12. *For* " standiug " *read* " standing."

Page 12. Lines 6 and 7. *For* the words " An elliptic...a *pseudo-elliptic integral* " *read* " An elliptic integral which is expressible either as the logarithm of an algebraic function of the variable or as such a logarithm plus another algebraic function of the variable is called a *pseudo-elliptic integral*."

Page 13. The first article should be numbered 13A and *not* 13.

Page 22. In the first line of Art. 25, *substitute* the word " frequently " for " generally." The Greek letter theta, in one form or another, is used to denote theta functions. The form θ used here is almost invariably found in French and Italian publications. For more information on theta functions, see Appendix B.

Page 32. Line 13. *For* " g_2 " *read* " g_3."

Page 38. Line 8. *For* " $c\omega'$ " *read* " $c\omega$."

Page 54. Line 1. *For* " Sielvek " *read* " Siebeck."

Page 54. Line 2. *For* " 54 " *read* " 57."

Page 57. Line 15 of the foot-note—
For " began " *read* " begun."

Page 58. Line 11. *For* "$\Phi(y_1)+\Phi(x_2)$" *read* "$\Phi(y_{\overline{1}})+\Phi(y_2)$."

Page 78. Line 7. *For* "$\left(\frac{e^{2\pi iz}}{\omega}\right)^n$" *read* "$\left(e^{2\pi iz/\omega}\right)^n$."

Page 82. First column of the table. *For* "1" *read* "θ_1"

Do. *For* "2" *read* "θ_2"

Do. *For* "3" *read* "θ_3"

Page 83. Line 9. *For* "22" *read* "21."

Page 90. Line 4. *For* "ω_1^2" *read* "ω^2."

Page 91. Theorem (XI). *For* "c_2" *read* "c_2^2."

Page 95. Line 2 of the foot-note. *For* "$Q \cos 2\pi v_1$" *read* "$Q \cos 2\pi v_2$.'

Page 96. Line 6. *For* "$\partial\ v$" *read* "∂v_ν."